LINEAR AND QUADRATIC EQUATIONS

1550-1660

SISTER MARY THOMAS à KEMPIS KLOYDA

of the

Sisters of Saint Francis

Rochester, Minnesota

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Preface

In the progress of science the development of the literal symbolism of algebra plays a role of fundamental importance. The literal symbolism makes possible for algebra itself not only general solutions but the fundamental developments connected with symmetric functions of the roots and with invariants and covariants.

The analytic geometry is not possible without some form of systematic symbolism; the use of letters lends itself admirably, particularly by the use of the devices of subscripts and primes to the widest extensions of the analytic geometry. At the same time the literal symbolism adapts itself immediately to the ideas involved in the differential and integral calculus. To the trigonometry the use of the literal symbolism as applied to the angles and to functions of the sides and other geometric lines was as vital as to the analytic geometry. At the same time the way was prepared for the introduction of the complex numbers and that more intimate understanding of the relationship between trigonometry and geometry, on the one hand, and algebra on the other, connected with the ideas of complex numbers and trigonometric theorems such as Demoivre's theorem.

So far as the application of mathematics to problems in physics is concerned the calculus, with its diverse forms of literal symbolism, presents the tools so essential to any fundamental control and understanding of the world in which we live.

The tremendously rapid progress of man's control over the physical universe after the period of Vieta, Descartes, Leibniz, and Newton is far more intimately connected with these achievements than is commonly assumed. The application and extension of the algebraic symbolism to varied problems of the physical universe followed closely upon the mathematical developments since the symbolism so readily lent itself to this use.

For the beginnings of the idea of a literal mathematical symbolism one turns naturally to the progress of algebraic ideas in Europe after the introduction of printing. This was the period during which the linear and quadratic equations became a universal part of elementary mathematics. At the same time very early in the sixteenth century numerical algebraic solutions of the cubic and biquadratic equations were formulated. In a sense one reaches an end of elementary algebra.

The study of the algebra as it developed in this period makes it clear that Vieta was led to the algebraic symbolism in connection with his attempt to formulate clearly and concisely the achievements made by Scipione del Ferro, Cardan, Tartaglia and Ferrari. This group of Italian mathematicians elaborated and extended the algebra of the Hindus and the Arabs by somewhat formal solutions of the cubic and biquadratic equations. Vieta, in presenting this material from a new point of view, made, almost incidentally, the final steps in the creation of a new algebraic symbolism.

The study here presented was made over a period of some years. In a measure every available book on algebra published up to 1660 has been searched for information about the progress of the elementary algebra. The study shows that Vieta made the final step, as noted, in the construction of a literal symbolism and shows further that the way for that step had been well prepared by a host of writers on algebra.

No small part of the labor involved in the present study is concerned with the location of copies of books which treat algebra. The student of the history of algebra may well set himself the task to examine and study all the texts on algebra

up to 1660. Where shall the student look for the list of these texts, and having the list where shall the student turn for the books themselves?

Fortunately at the University of Michigan efforts have been made over a period now of more than fifty years to acquire the early textbooks of mathematical science. With the start of an excellent collection of the early algebras the notable list of algebras presented in this work became a possibility. The generous assistance of libraries of America and Europe made it possible, also, to examine the great majority of the books themselves.

This work should be regarded by the reader as something more than a contribution to our knowledge of elementary algebra. In truth the material here presented is fundamentally a contribution to the history of mathematical ideas and their application to the physical universe.

Louis Charles Karpinski

University of Michigan
May, 1937

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The writer is grateful to the Graduate School of the University of Michigan for a gift from the Research Fund which made possible the securing of photostats of books inaccessible to the writer.

Sister Mary Thomas à Kempis Kloyda

Winona, Minnesota
September 12, 1937

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Chapter I

INTRODUCTION

The sixteenth and seventeenth centuries, with their profound changes in human endeavor, constitute a period of absorbing interest for the mathematician. To those who, from an historical point of view, regard the scientific achievements of present age as the most characteristic elements of modern civilization, these centuries are significant.

The restoration and adaptation of the science of antiquity were made possible by the work of translators who introduced it into the West. The consequent expansion of algebra took place during the course of the twelfth and thirteenth centuries and the subsequent period. It is toward a history of such an evolution that the present work is meant to offer a contribution.

The specific aim is to enable students to know as exactly as possible the status of the knowledge of simple and quadratic equations in Europe in the period 1550-1660. It is true that BORTOLOTTI, CAJORI, GLAISHER, TROPFKE and other historians have given scholarly consideration to the contributions of outstanding mathematicians of the fifteenth and sixteenth centuries. TROPFKE, in particular, has made an excellent study of quadratic equations covering a period of one thousand years. He, however, discussed only the more important discoveries of men of genius. Less brilliant, but significant, contributions were made by mathematicians of minor rank and, in this study, an attempt has been made through the investigations of their publications, to weigh their contributions, giving credit to all those who, within the period, furthered the evolution of mathematics.

To write a unified narrative of these results within the scope of such a work without making it a mere recital of names and contributions proved a difficult task. Consequently, the subject matter was divided into such topics as might enter into a logical development in the teaching of equations; namely, definition, formulation and classification of equations, symbolism and terminology, reduction of simple linear, simple simultaneous and quadratic equations, geometrical demonstrations and typical problems. Authors were quoted freely to illustrate the point under discussion. Errors found in histories and dictionaries were carefully noted.

It will be observed in the course of this study that a dividing line might be drawn between the works which precede the year 1630 and those which follow. A close analysis has revealed forces at work beyond 1630 which were different from those which resulted in the achievements of the earlier period.

The second purpose of the study of simple and quadratic equations is to present a check list of all the printed treatises dealing with algebra which appeared in the sixteenth and first half of the seventeenth centuries. A bibliography was compiled after an investigation of the most important libraries in America and Europe, including seven of the outstanding libraries in the Netherlands. This was made especially possible by the unusual collection of early algebras in the University of Michigan, containing, for example, as many algebras by Dutch authors of this period as were listed by the seven libraries just mentioned. These were supplemented by the works generously loaned by George A. Plimpton (GAP), the library of Columbia University (Col. U.), the library of Harvard University (HUL), the library of the Chicago University (LCU), and the University of Groningen (Holland) (UG), and photostats of books from the New York Public library (NYP), the Hispanic Society (HSA), the Huntington library (HN), the British Museum (BM), the Bibliothèque Nationale (Bibl. Nat.) and others. References to these libraries will be made by the abbreviations indicated above.

The bibliography is intended to be as comprehensive as possible. In this study, it has been necessary, in general, to limit actual examinations to one work of each author cited. This includes either the text itself or photostats of the text.

In the case of Gosselin, in lieu of De Arte Magna, which could not be secured, the investigation was confined to the excellent discussion of GOSSELIN by ENESTRÖM in the Bibliotheca Mathematica.

About one half of the books thus examined were written in Latin and the remainder in the vernacular. The most interesting fact, however, lies in the number of authors produced by the various countries. A survey reveals about nine Spanish authors, twelve English, nineteen each for Italy and France, and twenty-four each for Germany and the Netherlands.

The great number of texts published by the Dutch is worthy of comment. The geographical and economic conditions in the Netherlands explain the unusual mathematical activity. The geographical boundaries and governments were constantly changing. During the sixteenth century the Netherlands comprised Holland and Belgium, one part of which was under Spanish rule, the other under the French rule. This political arrangement certainly contributed not a little to the acquisition of mathematical knowledge. Furthermore, conditions were very favorable to commerce, as is shown through a critical study of contemporary arithmetics. In this particular period intellectual pursuits in the Netherlands touched every field of thought. The great number of Dutch writers in mathematics is undoubtedly the result of the stimulation of their natural ability by commercial contacts.

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Chapter II

SIMPLE AND QUADRATIC EQUATIONS BEFORE THE ADVENT OF PRINTING

In order to appreciate the development of algebra, especially in relation to simple and quadratic equations as found in books printed during the period just preceding 1573, the advent of PEREZ DE MOYA, to 1660, when MOORE first published his famous Arithmetic, it is necessary to consider briefly the different influences which have contributed to the various stages of its growth.

1. THE BABYLONIANS:

Modern scholars had, up to a few years ago, only a superficial knowledge of the advancement in mathematics made by the Babylonians. Recent discoveries have led historians to believe that the Babylonian-Sumerian mathematical ideas may bear some relation to those developed by the Arabs, Hindus, Egyptians and Greeks, though the exact nature and extent of this relationship has not yet been fully ascertained.

NEUGEBAUER and THUREAU-DANGIN have published a whole series of problems dealing with quadratic equations originating in geometrical problems which were solved by the Babylonians and Sumerians. One of the problems reads as follows:[1]

Given the area of a rectangle together with the difference of its sides, as 3:3 or 183, (being written in the sexagesimal system), and given the sum of the sides as 27, what is each of the sides?

In modern symbolism:

$$\begin{aligned} xy + x - y &= 183 \\ x + y &= 27 \end{aligned}$$

$$27x - x^2 + 2x - 27 = 183$$

$$x^2 + 210 = 29x$$

The solution of this is 15 for x and 12 for y. There is a second solution, with x = 14 and y = 13. In the same reference PROFESSOR KARPINSKI mentions that both solutions are found in the ancient cuneiform tablet preserved in the Louvre in Paris.

2. THE EGYPTIANS:

The study of the development of mathematics among this people draws attention primarily to the Ahmes Papyrus which contains a section devoted to the first-degree equation. The best-known problem reads thus:

A heap (unknown, hau) and its seventh makes 19.

The root $16\frac{1}{2}\ \frac{1}{8}$ is arrived at by an unmistakable and complete process of solution which can be applied equally well to any equation of the first degree in one unknown.

The same method of solution for simultaneous quadratic equations leading to pure quadratics of the type $ax^2 = n^2$ is found in the Kahun and Ilahun documents. It is noteworthy that the quadratics arise from the consideration of a problem in geometry which requires that an area of hundred square units be distributed into 2 squares whose sides have to each other a given ratio.

3. THE GREEKS:

(a) GREEK ANTHOLOGY: The Rhind Papyrus contains many simple equations which are similar to the arithmetical epigrams in the Greek Anthology, most of which appear under the name of METRODORUS (c. 491). "They are obviously only collected by Metrodorus, from ancient as well as more recent sources."[2] "Then about the time of DIOPHANTUS they became very popular and doubtless acted as a stimulus to his interest in algebraic problems,"[3] since many of them led to simple linear equations and still others to simultaneous equations of two or more unknowns. A few of these epigrams have been incorporated in the texts of Gemma Frisius (1549), Scheubel (1550), Peucer (1556),

[1] L. C. Karpinski, "The Mathematics of the Orient," School Science and Mathematics, May, 1934, p. 468.
[2] T. L. Heath, Greek Mathematics, 1921, Vol. II, p. 441.
[3] F. Cajori, A History of Mathematics, 1919, p. 59.

Lauremberg (1643) and others.

(b) PYTHAGORAS: Pythagoras himself has been credited by Plutarch with the discovery of the problem of the application of areas. It is equivalent to the geometrical solution of a mixed quadratic equation and, consequently, forms the essential part of what Heath has appropriately called geometrical algebra.

That the theory of the application of areas originated at least with the Pythagoreans we have on the authority of Eudemus, whom Proclus quotes as follows:

> These things are ancient, being discoveries of the Muse of the Pythagoreans, I mean application of areas (παραβολὴ τῶν χωρίων), their exceeding (ὑπερβολή) and their falling short (ἔλλειψις). It was from the Pythagoreans that later geometers (i.e., Apollonius of Perga) took the names, which they then transferred to the so-called conic lines (curves), called one of these a parabola (application), another an hyperbola (exceeding), and a third an ellipse (falling short), whereas those god-like men of old saw the things signified by these names in the construction, in a plane, of areas upon a finite straight line. For, when you have a straight line set out, and lay the given area exactly alongside the whole of the straight line, it is said to exceed, and, when you make it less, in which case after the area has been drawn there is some part of the straight line extending beyond it, it is said to fall short. Euclid, too, in the sixth book speaks in this way both of exceeding and falling short.[4]

It will be seen later what use was made of similar terms, for example, parabola and hyperbola, in the sixteenth century.

(c) EUCLID: Euclid incorporated into his geometry practically all accumulated knowledge of the fundamentals of elementary geometry. In consequence of this he wielded an influence which is still an enduring and unwaning reality.

The whole of Book II of Euclid's Elements constitutes an essential part of the Geometrical Algebra, which, as noted above, took the place of our algebra in Greek geometry. The first ten propositions give the equivalent of the following algebraical identities:[5]

1. $a(b + c + d \text{ -----}) = ab + ac + ad + \text{ ---}$
2. $(a - b)a - (a - b)b = (a + b)^2$
3. $(a + b)a = ab + a^2$
4. $(a + b)^2 = a^2 + b^2 + 2ab$
5. $ab + \{\frac{1}{2}(a + b) - b\}^2 = \{\frac{1}{2}(a + b)\}^2$ or $(\alpha + \beta)(\alpha - \beta) + \beta^2 = \alpha^2$
6. $(2a + b)b + a^2 = (a + b)^2$ or $(\alpha + \beta)(\alpha - \beta) + \alpha^2 = \beta^2$
7. $(a + b)^2 + a^2 = 2(a + b)a + b^2$ or $\alpha^2 + \beta^2 = 2\alpha\beta + (\alpha - \beta)^2$
8. $4(a + b)a + b^2 = \{(a + b) + a\}^2$, or $4\alpha\beta(\alpha - \beta)^2 = (\alpha + \beta)^2$
9. $a^2 + b^2 = 2\left[\{\frac{1}{2}(a + b)\}^2 + \{\frac{1}{2}(a + b) - b\}^2\right]$, or $(\alpha + \beta)^2 + (\alpha - \beta)^2 = 2(\alpha^2 + \beta^2)$
10. $(2a + b)^2 + b^2 = 2\{a^2 + (a + b)^2\}$, or $(\alpha + \beta)^2 + (\beta - \alpha)^2 = 2(\alpha^2 + \beta^2)$

Propositions 5 and 6 of this same book enable one to solve quadratic equations of the types:[6]

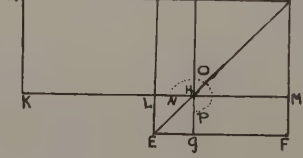
1. $ax - x^2 = b^2$ or $x + y = a$, $xy = b^2$

2. $ax + x^2 = b^2$ or $y - x = a$, $xy = b^2$

The most general form of the problem involving application with exceeding or falling short is found in Euclid's Elements VI. 28, 29. This is equivalent to the geometrical solution of quadratic equations.

One feature of Book II is the use of gnomon as found in particular in propositions 5 to 8. This is connected with the application of areas, and is therefore Pythagorean. Oughtred makes frequent use of the gnomon. Proposition 5 of EUCLID explains the meaning of this term:[7]

> If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.



Suppose that AB = a, DB = x;
then $ax - x^2$ = the rectangle AH = the gnomon NOP.

[4] T. L. Heath, Greek Mathematics, 1921, Vol. I, p. 150.
[5] T. L. Heath, Greek Mathematics, 1921, Vol. II, p. 379.
[6] Ibid., p. 380.
[7] T. L. Heath, Thirteen Books of Euclid, 1908, Vol. I, p. 382.

Thus, if the area of the gnomon is given (= b^2, say), and if a is given (= AB), the problem of solving the equation

$$ax - x^2 = b^2$$

is, in the language of geometry, To a given straight line (a) to apply a rectangle which shall be equal to a given square (b^2) and shall fall short by a square figure, i.e., to construct the rectangle AH or the gnomon NOP.

HEATH points out the exact correspondence between Euclid's geometrical method and the ordinary method of solving the equation. In order to avoid introducing a constant dependent on the sine of the angle of the parallelogram, he considers the parallelogram to be rectangular. The algebraic solution of

$$\frac{b}{c}x^2 - ax = -S$$

is $$x = \frac{c}{b}\cdot\frac{a}{2} \pm \sqrt{\frac{c}{b}\left(\frac{c}{b}\cdot\frac{a^2}{4} - S\right)};$$ [8]

Euclid arrives at the same result in only one root, namely the one with the negative sign before the radical. Heath explains that it was not customary for Euclid to give separate demonstrations for the other cases though the contrary has been implied by certain authors, as HEIBERG. After one case has been given the others are left to the students.

Commentaries on different phases of the works of EUCLID were written by Clavius, Peletier, Vieta, and Wallis. Many authors, Renaldini in particular, used Euclid's propositions to illustrate solutions in their own texts. It was, however, the period immediately following the one under consideration, 1550-1660, that the study of Greek mathematics and Euclid especially, held sway in England.

(d) ARCHIMEDES: Among this group of philosophers and mathematicians, one contributed to the development of algebraic ideas, Archimedes (278-212 B.C.). HEATH writes:

The fundamental property of the central conics appears in the works of Archimedes as a proportion $y^2:(a^2 + x^2) = b^2:a^2$ which, however, is equivalent to the Cartesian equation referred to axes with the center at the origin.[9]

The problem of the crown has interested many mathematicians and has, as a consequence, found its way into the texts of Aurel (1552), Peletier (1554), Peucer (1556), Antich Rocha (1565), De Billy (1643), Lauremberg (1643), Coutereels (1658), as well as into the Mathematical Recreations (1653) by Oughtred.

4. THE ALEXANDRIAN SCHOOL:

(a) APOLLONIUS OF PERGA: The "great geometer" (247-221 B.C.) as Apollonius was called, studied with the successors of Euclid at Alexandria. The legacy which his genius transmitted to posterity is the Conic Sections. PAPPUS mentions six other works which were supposed to have formed a part of the Treasury of Analysis, only one of which survived, namely the Sectio Rationis, or the "cutting-off of a ratio." Another treatise was On Determinate Section, in two books.

One general problem can be stated as follows:[10]

Given four points A, B, C, D, on a given straight line, to determine another point (P) on it such that AP.CP is to BP.DP in a given ratio.

The determination of P by means of an equation was not a difficult matter for the Greeks, since they were able to reduce it to the usual application of areas.

It is known that if one takes p as the parameter and d as the corresponding diameter, the properties of curves are the equivalent of the Cartesian equations referred to the diameter and the tangent at its extremity as axes (in general oblique).

$$y^2 = px \text{ (the parabola)}$$

$$y^2 = px \pm \frac{p}{d}x^2 \text{ (the hyperbola and ellipse respectively)}$$

Thus Appollonius expresses the fundamental property of the central conics as an equation between areas.[11]

[8] T. L. Heath, Thirteen Books of Euclid, 1908, Vol. II, p. 260.
[9] T. L. Heath, Greek Mathematics, Cambridge, 1921, Vol. II, p. 139.
[10] T. L. Heath, A Manual of Greek Mathematics, Oxford, 1931, p. 366.
[11] T. L. Heath, Greek Mathematics, 1921, Vol. II, p. 139.

(b) DIOPHANTUS (c. 250 A.D.) was one of the "last and most fertile mathematicians of the second Alexandrian School."[12] SMITH says of him that

> We cannot explain how it happened that he appeared like a giant in a century of pigmies.[13]

The influence that Diophantus exerted upon the development of algebra among the Arabs is inestimable. In 1575 XYLANDER (Wilhelm Holzmann) issued at Basel the first Latin edition of his works, which contained his Arithmetica, one of the first treatises devoted almost entirely to algebra. As a consequence, DIOPHANTUS is often considered "the father of this science."[14] He employed a certain algebraic symbolism, using a kind of sign for equality and another for the unknown quantity.

His first book contains problems leading to equations of the first degree. The following is an example of such:[15]

> To divide a given number into two parts having a given difference. Given number 100, given difference 40. Lesser number required x. Therefore
>
> $$2x + 40 = 100$$
> $$x = 30$$
>
> The required numbers are 70, 30.

PROFESSOR KARPINSKI states:[16]

> No doubt whatever exists that this Alexandrian was familiar with the analytical solution of the various forms of quadratic equations, neglecting negative roots and, of course imaginary roots, which did not receive serious treatment for more than a millenium after Diophantus. The three types of complete quadratic equations, involving only positive coefficients, are the following:
>
> $$ax^2 + bx = c$$
> $$ax^2 + c = bx$$
> $$ax^2 = bx + c$$
>
> These equations were treated only as contingent to the solution of problems.

Diophantus acquaints us with his intention of treating quadratics systematically when he explicitly states:[17]

> If the same powers of the unknown occur on both sides, but with different coefficients, we must take like from like until we have one single expression equal to another. If there are on both sides, or on either side, any terms with negative coefficients, the defects must be added on both sides until the terms on both sides have none but positive coefficients, when we must again take like from like until there remains one term on each side.
>
> This should be the object aimed at in framing the hypothesis of the propositions, that is to say, to reduce the equations, if possible, to one term. But afterwards I will show you how, when two terms are left equal to one term, such an equation is solved.

It is unknown whether he was able to accomplish this.

The first type of equation is illustrated by problem VI. 6 where it is required to find the right-angled triangle the sum of whose area and one of the perpendiculars makes a given number.

> Given number 7, triangle (3x, 4x, 5x). Therefore
>
> $$6x^2 + 3x = 7.$$
>
> In order that this might be solved, it would be necessary that (half the coefficient of x^2) product of the coefficient of x^2 and absolute term should be a square; but $(1\frac{1}{2})^2 + 6.7$ is not a square.
>
> Hence we must find, to replace (3, 4, 5) a right-angled triangle such that $(\frac{1}{2}$ one perpendicular$)^2$ + 7 times area = a square. Let one perpendicular be m, the other 1. Therefore $3\frac{1}{2}m + \frac{1}{4}$ = a square, or 14m + 1 = a square. Also, since triangle is rational, $m^2 + 1$ = a square. The difference $m^2 - 14m = m(m - 14)$; and putting, as usual, $7^2 = 14m + 1$, we have $m = \frac{24}{7}$. The auxiliary triangle is therefore $(\frac{24}{7}, 1, \frac{25}{7})$ or (24, 7, 25). Starting afresh, we take as the triangle 24x, 7x, 25x. Therefore,

[12] F. Cajori, A History of Mathematics, 1919, p. 60.
[13] D. E. Smith, History of Mathematics, 1923, Vol. I, p. 135.
[14] Hankel, Geschichte der Mathematik, 1874, p. 158.
[15] T. L. Heath, Diophantus of Alexandria, 1910, p. 131.
[16] L. C. Karpinski, Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi, 1915, p. 7.
[17] T. L. Heath, Diophantus of Alexandria, 1910, p. 131.

$$84x^2 + 7x = 7$$
$$x = \tfrac{1}{4}.$$

We have then $(6, \frac{7}{4}, \frac{25}{4})$ as the solution.[18]

The correspondence between this problem and that of some of the Babylonian quadratics is readily perceived.

Diophantus used the symbol ς for our x and "one unit," i.e., 1, for our y. After the value of the first unknown was obtained, he substituted this value for "1" and thus found the value of the second. This use of the "unit" was imitated by later writers.

The "double-equation," as used by Diophantus, requires that one find a value of the unknown quantity which will make two different functions of it rational squares. No further discussion of this type of equation will be given here.

Heath comments on the fact that Diophantus gave no account of more than one root, even when both roots are positive rational numbers. Thus he explains that Diophantus desired to find some rational solution and, consequently, when he obtained such a one, he proceeded no further to solve for the other, even though he knew that it existed.

5. THE HINDUS:

(a) ARYABHATA and BRAHMAGUPTA: One of the earliest and most prominent Hindu mathematicians was born in 476 A.D., some two centuries after the period of Diophantus. RODET presents the work of ARYABHATA in Leçons de Calcul d'Aryabhata, Journal Asiatique, seventh series, Vol. VIII (1879), pp. 393-434. It is here that the solution of quadratic equations is embodied in the rule for finding the terms of an arithmetical series as WALTER E. CLARK demonstrates on page 36 of his The Aryabhatiya of Aryabhata. A second reference to quadratic equations is made on pages 38-9 of the same study in problem 25, which reads as follows:

Multiply the sum of the interest on the principal and the interest on this interest by the time and by the principal. Add to this result the square of half the principal. Take the square root of this. Subtract half the principal and divide the remainder by the time. The result will be the interest on the principal.

The formula involves the solution of a quadratic equation in the form of $ax^2 + 2bx = c$.

A sum of money is loaned. After a certain unit of time the interest received is loaned for a known number of units of time at the same interest. What is known is the amount of the interest on the principal plus the interest on this interest. Call this B. Let the principal be A. Let t be the time.

$$x = \frac{\sqrt{B \times A \times t + (\frac{A}{2})^2} - \frac{A}{2}}{t}.$$

The following example is given by Parameswara. The sum of 100 is loaned for one month. Then the interest received is loaned for six months. At that time the original interest plus the interest on this interest amounts to 16.

B = 16
A = 100
t = 6

$$\frac{\sqrt{16 \times 100 \times 6 + 2500} - 50}{6} = 10.$$

The interest received on 100 in one month was 10.

This same author explains that a method for the solution of indeterminate equations of the first degree is only briefly given in Aryabhata. Aryabhata was not, however, the only student of analysis in India. Ujjain, the center of Indian learning, presents BRAHMAGUPTA as the author of an algebra in the early part of the seventh century. This algebra contains a rule for solving quadratic equations as follows:[19]

To the absolute number multiplied by the (coefficient of the) square, add the square of the half of the (coefficient of the) unknown, being divided by the (coefficient of the) square, is the unknown.

In formula this corresponds to the solution

$$x = \frac{\sqrt{(\frac{b}{2})^2 + ac} - \frac{b}{2}}{a}$$

of the equation $ax^2 + bx = c$.

(b) MAHAVIRACARYA and BHĀSKARA: Contemporary with Al-Khowarizmi is the Hindu

[18] T. L. Heath, Diophantus of Alexandria, 1910, p. 228.

[19] L. C. Karpinski, Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi, 1915, p. 10.

writer MAHAVIRACARYA, whose arithmetical and algebraic work has been translated into English by M. RANGCARYA. The work is significant in that it shows a continued interest in algebra in India from the time of Aryabhata. One historian notes a distinguishing characteristic of his work as follows:[20]

> A peculiarity of treatment is that the unknown quantity and the square root of the unknown appear rather than the unknown and its square.

Three centuries later BHĀSKARA (b. 1114 A.D.), another Hindu mathematician, made important contributions to the advance of this science.

(c) AL-KHOWARIZMI: The first systematic treatise on algebra was the product of an Arabian writer, Mohammed ibn Musa, Al-Khowarizmi (c. 825 A.D.). So important were his writings on arithmetic that, just as "Euclid" is in England a synonym for elementary geometry, so the word "algorism," the patronymic of Al-Khowarizmi, designated for a long time in Europe our present system of arithmetic. The word "algebra" is a part of the title, al-jebr w'almuqabula, of his treatise on the subject. The algebra of Al-Khowarizmi is not a simple translation of the Hindu work; it is rather the work of a scientist who has profited both by his study of Greek and Hindu mathematics.[21]

One linear equation and five types of quadratic equations are found in his work.

$$ax^2 = bx, \qquad ax^2 = n, \qquad bx = n$$

$$ax^2 + bx = n, \quad ax^2 + n = bx, \quad ax^2 = bx + n$$

The word "algebra," meaning "restoration," refers to the transfer of the negative term in an equation; thus the operation of changing $x^2 - 10x = 21$ to $x^2 = 10x + 21$ is the operation of "algebra," the x^2 being as incomplete of an amount 10x, which is restored to it by the process. Complete recognition of positive and negative quantities, with the simple type form of the quadratic, $ax^2 + bx + c = 0$, did not come until after Vieta. The problems are given, of course, with numerical coefficients; the analytical procedure of the solution for each type would apply to any problem of the type. To the analytical solution is added a proof by this use of a geometrical figure involving propositions largely of the second book of Euclid.[22]

In order to acquaint the reader with Al-Khowarizmi's method of solving the types of equations mentioned above, the fifth problem illustrating the fifth chapter, as found in the scholarly work on Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi has been selected.

> I divide ten into two parts in such a way that tne sum of the products obtained by multiplying each part by itself is equal to 58.
>
> Explanation. You multiply 10 - x by itself, giving 100 and x^2 less 20x. Then multiply x by itself, giving x^2. Collecting the products of these two multiplications you obtain 100 and $2x^2$ less 20x equal to 58. Complete the 100 plus $2x^2$ less 20 x by adding 20x to 58. This gives 100 units + $2x^2$ equal to 58 units and 20x. Now you reduce this to one square, and then by opposition (This operation corresponds to the Arabic term Almuqabala. In this instance the 29 on the right balances or cancels an equal amount of the 50 on the left). You take 29 from 50, leaving $21 + x^2$ equal to 10x. Therefore you halve the roots, giving 5. Multiply this by itself, giving 25. From this you subtract 21 and 4 is left. Take the root of this, 2, and subtract it from 5, i.e., from the half of the roots. This gives 3 which represents, of course, one part in ten. So this problem has led you to the fifth of the six chapters in which we treated the type, a square and a number equal to roots.[23]

(d) ABU KAMIL: The second in the point of time of the great Arabian writers in this field is Abu Kamil Shoja' ben Aslam who wrote in the ninth or early tenth century a much more extensive treatise on algebra than that of Al-Khowarizmi.[24]

In the opening sentence of his Algebra he refers to the species of Al-Khowarizmi, namely, roots, square and number, and also to his types of equations, squares equal to roots, squares equal to numbers and roots equal to numbers ($ax^2 = bx$, $ax^2 = n$, $bx = n$). He also refers to Euclid, but aside from these two names no others are noted in his work. He then goes on to say that we "construct now these three species, that is, roots, squares and numbers, joined by twos and proportional to the third. This gives, a square and roots equal to numbers, a square and numbers equal to roots, and roots and numbers equal to squares." (In modern notations, $ax^2 + bx = n$, $ax^2 + n = bx$, $bx + n = ax^2$).[25]

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[20] L. C. Karpinski, Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi, 1915, p. 11.
[21] L. C. Karpinski, "The Origin and Development of Algebra," in School Science and Mathematics, Jan., 1923, Vol. XXIII, p. 59.
[22] Ibid., p. 60. [23] L. C. Karpinski, 1915, p. 107.
[24] L. C. Karpinski, "The Algebra of Abu Kamil" in The American Mathematical Monthly, February, 1914, Vol. XXI, p. 38. [25] Ibid., p. 39.

He asserts, further, that the first type of equation was given by Al-Khowarizmi. The example is as follows:

A square and 10 roots are equal to 39 drachmas ($x^2 + 10x = 39$) —— This is that you divide the roots in half, that is in two equal parts. In this question this gives 5. You multiply this by itself, giving 25, which you add to 39. This gives 64 of which number you take the root which is 8; from this you take half of the roots, that is, 5, and there remains 3, which is the root of the square. The square is 9.[26]

The second type of quadratic equation is also the same as Al-Khowarizmi's $x^2 + 21 = 10x$. Al-Khowarizmi, gave the geometrical figure corresponding only to the smaller root of the equation. Thus in the equation, $x^2 + n = bx$, in which

$$x = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - n}$$

one of the roots, when both are real, is necessarily less than $\frac{b}{2}$ and the other is greater. Abu Kamil gives the geometric explanation corresponding to both, thus completing the work of Al-Khowarizmi.[27]

In the third type of equation Al-Khowarizmi, in taking the equation, $3x + 4 = x^2$, gave three explanations for finding the root of the square. The algebra up to this point is summarized by stating that all questions that can be solved by algebra and almuchabala, that is, restoration and opposition, must reduce to one or the other of the six types of the equations mentioned above.[28]

The most important conclusion drawn by PROFESSOR KARPINSKI from this investigation of ABU KAMIL'S Algebra is that two prominent algebraists made extensive use of Abu Kamil's treatise without, however, giving him due credit. These two are AL-KARKHI (died c. 1029) and LEONARD OF PISA.

Al-Karkhi reproduced not only problems given by Abu Kamil, but also the geometrical solutions devised by this writer. In this same "fakhri," Al-Karkhi used Diophantine type of algebraic analysis, making noteworthy extensions of indeterminates of higher degree.[29]

(e) GHERARD OF CREMONA: Gherard of Cremona (d. 1187) is generally believed to have translated from the Arabic, many works of Greek authors, including the Elements of EUCLID. The result of these translations, together with others, like that of ROBERT OF CHESTER mentioned in connection with the study of AL-KHOWARISMI, is a profound and far-reaching interest in Greek and Arabian mathematics.

(f) LEONARD OF PISA (Fibonacci): One of the first results of this interest is found in the excellent works of Leonard of Pisa.

In the Liber abaci first published in 1202 and later in 1228 in an improved edition, he gave a fine treatise on arithmetic and algebra as known to the Arabs, incorporating a great number of problems of Abu Kamil and illustrations of Al-Khowarismi, but to some extent written in a free and independent style of his own. It would seem that this should have stimulated a great advance in the next centuries, but as Hankel says, when we look at the work of Luca Paciuolo nearly three centuries later, we find that the talent Leonardo left to the Latin world had lain hidden in a napkin and had earned no interest.[30]

HANKEL'S statement is not entirely true, for the work was a source of inspiration to many intermediate writers in Italy.

Since Leonard derived his work entirely from Arabic sources, he used such terms as res or radix for the unknown quantity, census for its square, and the like.

On fol. 187^v of his Liber abaci he explains the types of equations that he employs,

he autem in solutionibus questionum inter se equantur sex modis, ex quibus tres sunt simplices, et tres compositi. Primus quidem modus est, quando quadratus, qui census dicitur, equatur radicibus; secundus, quando census equatur numero; tertius, quando radix equatur numero.

[26] L. C. Karpinski, "The Algebra of Abu Kamil" in The American Mathematical Monthly, February, 1914, Vol. XXI, p. 40.

[27] Ibid., p. 42.

[28] Ibid., pp. 43-44.

[29] L. C. Karpinski, "The Origin and Development of Algebra," in School Science and Mathematics, January, 1923, Vol. XXIII, p. 61.

[30] T. L. Heath, Greek Mathematics, 1921, Vol. I, p. 367.

PROFESSOR KARPINSKI has compared very carefully the problems of Leonardo and those of Al-Khowarismi and finds that the second problem presented by the former in the section heading: Expliciunt introductiones algebre et almuchabale. Incipiunt questiones eiusdem leads to the first problem given by the latter, in which ten is divided into two parts such that the larger part should equal $1\frac{1}{2}$ times the product of the two parts. There are many such similarities between these two authors, thereby demonstrating the extent of the influence of Al-Khowarizmi.

Chapter III

SIMPLE AND QUADRATIC EQUATIONS
FROM THE INVENTION OF PRINTING TO 1550

The period revolving around the invention of printing was one in which Europe expressed an eager desire for more learning. In pursuit of learning, attention was concentrated on ancient thought, especially that from Greek sources. As a consequence, the twelfth, and to some extent, the thirteenth century, became a period of translations. Though the translators came from all parts of Europe, the most influential among them were ROBERT OF CHESTER and GERARD OF CREMONA.

The immense activity of the translators proved that there was a waxing demand in the West for mathematical knowledge and mathematical books. This could not fail to stimulate any genius that might exist, and that is the very thing that happened. Fibonacci and Nemorarius stand at the threshold of European mathematics. Fibonacci's Liber abaci of 1202 may be called the first monument of European mathematics, whose author was the forerunner of a splendid and endless legion of western mathematicians.[31]

Thus, by the middle of the fifteenth century, the works of the Greek and Arabic mathematicians were made available to European scholars. The invention of printing facilitated the diffusion of their researches.

In a scholarly article,[32] the title of which clearly indicates its principal object, J. W. L. GLAISHER treats the most important writers who appeared immediately after the invention of printing; namely, Paciolo, Cardan, Riese, Rudolff, Grammateus, and Stifel. This author also illustrates the application of symbols by discussing problems, particularly those which led to equations. He covers the early period to 1550 adequately. Consequently, this critical study of quadratic and linear equations is confined to the works immediately following. Nevertheless, in order to appreciate fully the development of the subject after 1550, it is desirable to include a resumé of Glaisher's investigation, as well as notes on other algebraists who have contributed much to the development of algebra, notably Chuquet and Tartaglia, in the period 1478-1550.

1. CHUQUET (Estienne de la Roche):

In 1484 appeared Le Triparty en la science de nombres by Nicholas Chuquet, a physician in Lyons. CAJORI calls it "a manuscript of great originality" and states that "had Chuquet's work had been printed at the time when it was written, it would have accelerated the progress of algebra."[33] CANTOR asserts that the influence of Le Triparty was extended by its circulation and further accentuated by the publication of Larismethique by de la Roche at Lyons in 1520. Examination of this work shows that de la Roche did not copy the algebra of Chuquet's work, with the exception of some ideas which he introduces rather unexpectedly into his treatise. It is true, as Cajori says, de la Roche "did not fully enter into the spirit of his predecessor nor did he fully comprehend his work."[34] Cajori asserts that had the contrary been true, de la Roche, as a sympathetic interpreter, might have contributed no mean amount to the history of science. De la Roche exerted very little influence before the nineteenth century and none at all after that period. He is mentioned by Gosselin and Buteo, and later by John Wallis.

In the work of Chuquet, but not in that of de la Roche, the algebra is called the Rigle des premiers. On page 752 of volume XIII of BONCOMPAGNI'S Bolletino, CHUQUET gives the problem of finding two

[31] G. Sarton, Introduction to the History of Science, Baltimore, 1931, Vol. II, p. 7.

[32] "On the Improvements in Algebraical Notation Made by Stifel, with an Account of Some of the Notations Used by the Early Algebraists" in Messenger of Mathematics, Vol. LVI, p. 33—— Hereafter referred to as Glaisher, Improvements.

[33] Notations, Vol. I, p. 100.

[34] Ibid., p. 103.

numbers whose sum is 17 and difference is 3. He designated the large number by ".1." and the smaller one by ".17. $\bar{m}$ .1'.", which is subtracted from ".1'.", leaving ".1' $\bar{m}$ .17. $\bar{p}$ 1'." or ".2'. $\bar{m}$ .17. egaulz .3." The parts are now contracted so that one has ".2'." for one member and ".20." for the other. The author completes the problem by saying,

Ores partiz le nombre por les premiers si auras .10. pour lung des nombres et par sousequent .7. pour autre.

In modern notation the solution would be as follows:

$$\begin{aligned} x &= \text{lesser number} \\ 17 - x &= \text{greater number} \\ x - (17 - x) &= 3 \\ 2x - 17 &= 3 \\ 2x &= 20 \\ x &= 10, \text{ or the lesser number} \end{aligned}$$

$17 - x$ or $17 - 10 = 7$, the greater number.

He employed equations that can be classified under one or other of the general types so commonly used. For example, in one problem he obtains the equation $12 + 3x = 4x^2$ (type: $ax^2 = bx + c$). He gives the rule for its solution by directing the division of each term by four, making the coefficient of the middle term $\frac{3}{4}$, and the squaring of one-half of that number. This in turn is added

avec le precedent qui est .3. ainsi lon aura .3.$\frac{9}{64}$. dont .R^2.3.$\frac{9}{64}$. p.$\frac{3}{8}$. Qui est la moitie du moyen le nombre que lon suche.

Modern notation:

$$\begin{aligned} 4x^2 - 3x &= 12 \\ x^2 - \tfrac{3}{4}x &= 3 \\ x^2 - \tfrac{3}{4}x + \tfrac{9}{64} &= 3 + \tfrac{9}{64} \\ x &= \tfrac{3}{8} \pm \sqrt{3 + \tfrac{9}{64}}. \end{aligned}$$

In the original solution only one root was indicated.

Another problem whose equation is $12 + 3x^2 = 9x$, has imaginary roots which Chuquet characterizes:

Il senz que ceste rai est impossibile.

Cajori calls attention to the fact that this author "took a position in advance of his time when he computed with zero as if it was an actual quantity," obtaining, "$x = 2 \pm \sqrt{4 - 4}$ as the roots of $3x^2 + 12 = 12x$."[35]

In note 2 on page 413 of volume XIV, of Bolletino, Boncompagni makes this statement:

Simon Stevin est, peut-etre-le seul mathematicien qui est fait usage d'une denomination analogue à celle de Chuquet. Il donne en effet le nom de "premiere quantité" et de "Prime quantité" à ce que ses contemporains appellaient coste, res, cosa, racine cossique. Dans les problems qu'il resout par l'algebre, il represente l'inconnue, le "nombre requis" par le notation .1 (L'Arithmetique //De//Simon Stevin//De Bruges,//...

Reference to the study of notations in this work shows that others have used similar notations. As a consequence, it may be questioned whether STEVIN actually received his inspiration from Chuquet.

2. FIFTEENTH CENTURY MANUSCRIPTS:

GLAISHER in Part II of his Improvements[36] briefly summarizes in ten extracts the notations of the early German Algebraists before 1540 from the fifteenth century manuscripts in Codex 14908 of the Munich library and in Codex C 80 of the Dresden library.

In number III of the Munich manuscripts is an equation equivalent to $\sqrt{(x^2 - x)} + x = 2$, the solution of which resembles the modern method, and employs the terms numerus, zal, dragme, wurcz, zensus (or census), zins for the unknown. Some of the other manuscripts use ding or cosa. The first power of the unknown is variously represented by 1∂, co., ϑ and 2ϱ. This last sign, as shown in Karpinski's Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi, is the abbreviation for radix, "apparently formed from an r followed by a flourish; it can also be an abbreviation for res." x^2 was symbolized by ʒ , z, z^o, and x^3 by œ, "a c with a flourish,"[37] as Glaisher points out. ʒʒ stood for x^4, and so on.

[35] F. Cajori, Notations, Vol. I, p. 101.

[36] pp. 70-73.

[37] Glaisher, Improvements, f.n., p. 72.

Attention is called to these terms and symbols because of their possible influence upon authors of the sixteenth and seventeenth centuries critically studied in this work.

3. FIRST PRINTED ALGEBRA IN A SEPARATE TEXT:

In 1518 GRAMMATEUS (Heinrich Schreiber, d. 1525) published at Vienna his Ein new künstlich behend und gewiss Rechenbüchlin vff alle Kauffmannschafft ... which was "the first German book in which algebra was treated systematically."[38] Numerus is represented by N, x by la: or pri:, x^2 by 2a: or se:, x^3 by 3a: or ter:, abbreviated in some places to t, He explains that numerus linealis is called radix or prima quantitas, numerus superficialis is called quadratus, and numerus corporalis is called cubus. The equations are classified into seven types as follows:[39]

	Modern Notation
Die erst auffgab.	
Wann radix is (sic) gleich N.	$bx = N$
Die ander auffgab.	
Wann census is (sic) gleich N.	$ax^2 = N$
Die drit auffgab.	
Wann cubus ist gleich N.	$ax^3 = N$
Die vierdt auffgab.	
Wann census und radix sein gleich N.	$ax^2 + bx = N$
Die fünfft auffgab.	
Wann census und N sein gleich radici.	$ax^2 + N = bx$
Die sechst auffgab.	
Wann radix und N werden ergleicht cens.	$bx + N = ax^2$
Die sebind auffgab.	
Wann census von cens. wurt vergleicht N.	$ax^4 = N$

Under the first rule we find an example stated thus:

2 sext ist gleicht 4 quint or 2 pri. is gleich 4.N facit 1 pri. 2N.

In modern notation: $2x^6 = 4x^5$

or, $2x = 4$

$x = 2$.

This is followed by similar problems together with those dealing with two numbers in double proportion, and also by a merchant problem. These types of problems are met frequently later.

4. ADAM RIESE:

Adam Riese who was "the foremost arithmetician of Germany for a century,"[40] published an algebra in 1524. This, however, remained in manuscript form until 1860, when BERLET published an abridgment of it.

There is a striking similarity between Riese's names and symbols and those in Rudolff's Coss.

The eight classes of equations used by RIESE, include simple linear equations, pure quadratic, pure cubic and pure biquadratic equations together with the types, $ax^2 + bx = c$, $ax^2 - bx = c$, $ax^2 - bx = -c$, and $ax^2 + bx = -c$. The solution of the last type, he tells us, will be either negative or impossible.

5. CHRISTOFF RUDOLFF:

Christoff Rudolff, referred to above, was a pupil of Grammateus and "the writer of the first book on algebra in the German language."[41] This was published in 1525. The symbols employed are similar to those found in the early German manuscripts. It is in the fifth chapter of his Coss that algebra is introduced.

Glaisher, in Notation used in the Treatment of Equations by Rudolff, discusses "the need for a notation for a second unknown by Rudolff" as follows:

> The second unknown is first mentioned in Example 37 of the first rule (which concerns simple equations). In this example it is required to find the ratio of two lines ab and cd, which are such that if $\frac{1}{4}$ of cd is added to ab and $\frac{1}{8}$ of this sum is added to what is left of cd, then the sum is equal to what is left of ab.
>
> Rudolff points out that this example contains two 'hidden positions' (unknowns), so that, if ab be denoted by 1v(, something else must be put for cd. He says that he will explain later on how it is accomplished by the regula quantitatis 'which is the completion of the coss: yea truly such a completion that without it, it would be no more than a trifle.'
>
> As Rudolff so far has only one position at

[38] Glaisher, Improvements, f.n., p. 74.
[39] Grammateus, Ein new Künsteich Buech, Nürnburg, 1518.
[40] L. C. Karpinski, "Arithmetical Centenarian Textbooks" in Scripta Mathematica, November, 1933, p. 37.
[41] F. Cajori, A History of Mathematics, 1919, p. 140.

his disposal, he puts 1$\mathcal{N}$ for the length of ab, and as he must put something for the length of cd he takes it to be the number 12, which he writes as 12ϕ. He adds $\frac{1}{4}$ of 12ϕ to ab making $\mathcal{N}$ + 3ϕ, the remainder of cd being 9ϕ. Further, $\frac{1}{3}$ of 1$\mathcal{N}$ + 3ϕ is $\frac{1}{3}\mathcal{N}$ + 1ϕ; to this add 9ϕ; the sum is $\frac{1}{3}\mathcal{N}$ + 10ϕ; this is to be equal to what is left, that is, to $\frac{2}{3}\mathcal{N}$ + 2ϕ, whence ab is found to be 24ϕ. Thus it is double of cd.

Although two unknown lengths are involved, it is only their ratio that is required, so that Rudolff is quite justified in assigning to one of them an arbitrary number such as 12.

Glaisher points out that, in problems involving three unknowns, RUDOLFF employs the same procedure as Paciolo and that he states rules for the solution of the first and second-degree equations, including the three types of irrational quadratics.

6. PACIOLO:

The Summa de arithmetica, geometria proportioni et proportionalis, was written in 1494 by Luca Paciolo (1445-1514). Since this text was a compendium of the existing knowledge on arithmetic, algebra and trigonometry, it gives one an excellent survey of the state of mathematics in Italy in Paciolo's day, and of algebra especially. With the interest in mathematics that had developed in Europe at this time, such a work could not fail to exert untold influence. "The second edition (1523) is a posthumous publication and differs from the first edition only in the spelling of some of the words."[42]

Glaisher explains that,

> Neither in Leonardo's algebra nor in the fourteenth century manuscript printed by Libri is there any attempt to represent a second unknown, but in Paciolo's Summa a second unknown is used in the solution of several questions, and denoted by 1 quantita, the first unknown being 1 cosa; and Paciolo mentions that this is modern usage, the early writers call it a second cosa. The word quantita is contracted in various ways, both when it is used in its ordinary sense, and when in its technical sense, the shortest abbreviation being similar to qp^{a}, if a single downstroke is made to serve for both p and q. For convenience of printing I write this abbreviation ϕ^{a}.[43]

Plus and minus (in Italian piu and meno) are indicated by $\bar{p}$ and $\bar{m}$. The modern x is represented by co. (cosa), and censo means x^2. These are similar to the symbols already referred to. In Paciolo's Summa a second unknown is denoted by 1 quantita, which Glaisher writes as ϕ^{a}, to distinguish it from the first unknown, which is 1 cosa. Paciolo mentions that "this is modern usage, the earlier writers having called it the second cosa."[43] The use of these terms and their abbreviations can best be illustrated in a problem wherein three quantities in continued proportion are such that the product of the first and third is 36, which is the square of the middle one. The sum of the three quantities is 216. Paciolo assigns 1 co. for the middle term and puts "1 co. - una quantita" for the first quantity and "1 co. + la ditta una quantita" for the third. He multiplied "1.co. vna ϕ^{a}" by "1.co. + $\bar{q}$lla $\ddot{\phi}$ tita" and obtains "1.ce. - 1.ce. de ϕ^{a}" as the product. Thus 1.ce. deϕ^{a} is equal to "1.ce. - 36". It is worthy of note that the modern use of transposition is implied. The result of this problem is $\sqrt{(1.ce. - 36)}$.[44]

Later on Paciolo symbolized quantita by a symbol, which was written as qp^{a} by Glaisher in the solution of simple simultaneous equations. These equations will be treated later in a comparative study of similar problems solved by Cardan.

One point that had not been noted by historians until Cajori drew attention to it, is Paciolo's use of the dash (-) as a symbol of equality. For example, 2.co. p 6-216 is the modern equivalent of $2x + 6 = 216$.

Paciolo divides quadratic equations into the three historical classes and demonstrates them by geometry. He, too, recognizes only positive roots, but he also recognizes the impossibility of square roots of negative numbers.

7. CARDAN:

Cardan (1501-1576) "in his Ars Magna gives us the first great Latin treatise

[42] F. Cajori, Notations, p. 106.
[43] Glaisher, Improvements, p. 40.
[44] Idem.

HIERONYMI
CARDANI MEDIO
LANENSIS, CIVISQVE BONO-
NIENSIS, PHILOSOPHI, MEDICI ET
Mathematici clarissimi,

OPVS NOVVM DE
PROPORTIONIBVS NVMERORVM, MO
TVVM, PONDERVM, SONORVM, ALIARVMQVE REBVM
mensurandarum, non solum Geometrico more stabilitum, sed etiam
varijs experimentis & obseruationibus rerum in natura, solerti
demonstratione illustratum, ad multiplices usus ac
commodatum, & in V libros digestum.

PRAETEREA
ARTIS MAGNÆ, SIVE DE REGVLIS
ALGEBRAICIS, LIBER VNVS, ABSTRVSISSIMVS
& mexhaustus plane totius Arithmeticæ thesaurus, ab
authore recens multis in locis recogni-
tus & auctus.

ITEM
DE ALIZA REGVLA LIBER, HOC EST, ALGEBRAICAM
logisticen sui, multorum recondita numerandi subtilitate, secundum Geo-
metricas quantitates inquirentis, necessaria Corollis,
nunc demum in lucem edita.

Opus Physicis & Mathematicis apprimè
utile & necessarium.

Cum Cæs. Maiest. Gratia & Priuilegio.
BASILEÆ.

Fig. 1. Title page of Cardan's Ars Magna (1570).

cubus a b cùm sexcuplo a b quod est cum sex rebus, nam a b est latus sui cubi, æquatur 20 igitur cū & b h cubus cum sexcuplo b h æquatur 20 erit b h cubus cum sexcuplo b h æqualis cubo a b cum sexcuplo a b, igitur a b est res, et ipsa est differentia duorum laterum producentium 2, & quorum cubi differunt 20 quod erat demonstrandum. Ex his conficiemus regulam.

24	1	25
20		13 7
4	14	18

REGVLA:

Deducto tertiam partem numeri rerum ad cubum, cui addes quadratum dimidij numeri æquationis, & totius accipe radicem, scilicet quadratam, quam seruabis, unius dimidium numeri quod iam in se duxeras, abijcies, ab altera dimidium idem minues, habebisq; Binomium cum sua Apotome, indè detracta ℞ cubica Apotomæ ex ℞ cubica sui Binomij, residuū quod ex hoc relinquitur, est rei æstimatio. Exemplum. cubus & 6 positiones, æquantur 20, ducito 2, tertiam partem 6, ad cubum, fit 8. duc 10 dimidium numeri in se, fit 100, iunge 100 & 8, fit 108, accipe radicem quæ est ℞ 108, & eam gemina bis, alteri adde 10, dimidium numeri, a b altero minues tantundem, habebis Binomium ℞ 108 p: 10, & Apotomen ℞ 108 m: 10, horum accipe ℞ cub^a & minue illam quæ est Apotomæ, ab ea quæ est Binomij, habebis rei æstimationem, ℞ b: cub: ℞ 108 p: 10 m: ℞ v: cubica ℞ 108 m: 10.

cub^s p: 6 reb^s æqlis 20	
2	20
8	10
108	
℞ 108	p: 10
℞ 108	m: 10
℞ v: cu. ℞ 108	p: 10
m: ℞ v: cu. ℞ 108	m: 10

Aliud. cubus p: 3 rebus æquatur 10: duc 1, tertiam partem 3, ad cubum, fit 1 duc 5, dimidium 10, ad quadratum, fit 25 iunge 25 & 1, fiunt 26, huius radici adde 5, & ab ea minue 5, habebis Binomium ℞ 26 p: 5, & Apotomen ℞ 26 m: 5, igitur rei æstimatio est ℞ v: cubica ℞ 26 p: 5 m: ℞ v: cubica ℞ 26 m: 5, experientia sic habetur:

ra: v: cubica ra: 26 p: 5	ra: ra: v: cubica ra: 27 m: 5
cubi partium ra: 26 p: 5 m:	ra: 26 m: 5

hoc autem totum, ut liquet, est 10.

Quadratum partium, ra: v: cubica 51 p: ra: 2900	ra: v: cub^a 51 m: ra: 2600
triplicata quadrata partium,	ra: v: cub: 1377 p: ra: 1895400
ra: b: cubica 1377 m: ra: 1654400	partes ipsæ
m: ra: v: cubica ra: 26 m: 5.	p: ra: v: cubica ra: 26 p: 5

Productū partium in triplicata quadratorum

p: ra: v: cubica 49299554 p: 6885 m: ra: 4738500611: 70 20
m: ra: v: cubica 49299554 m: 6885 m: ra: 4738500611: 70 20

Porro hę ra: cubicę quatuor nominibus constantes, ad duas reduci

Fig. 2. A page from the same text exhibiting Cardan's treatment of cubics.

formæ positio, & quia ut uides in figura tertiæ regulæ, el & m k, ... rum duplo g c in a b, & g c est aptitio, possum numerum quadratorum addendorū semper 2 positiones, id est duplū g c, & quia ... addendus ad 36, est l n m, & ideo quadratum g c cum eo ... duplo, hoc est ex g c duplicato in c b, seu ex g e in duplum e b, & est 12, ... crus quadratorum priorum, ducam igitur 1 positionem, dimidium numeri quadratorum additorum, semper in numerū quadratorum priorum, & in se, & fient 1 quadratum p: 12 positionibus addis ex alia parte, & etiam 2 positiones pro numero quadratorū, ... bemus igitur iterum ex communi animi sententia, quantitates inter se scriptas, inuicem æquales, & utraq; habent radicem, prima ex regula tertia, sed secunda quantitas ex supposito,

1 q̃d quad, p 02 pos̃ p: 12 q̃d ℞ p: q̃d. p: 12 pos̃. additi numeri p: 36 æqualia.	
2 pos̃ p: q̃ quadrato. p: 60 pos̃ p: 1 q̃d. p: 12 pos̃. numeri additi.	

igitur ducta prima parte trinomij in tertiam, fit quadratū dimidiæ partis secundæ trinomij, quia igitur ex dimidio secundæ in se, fiunt 900, quadrata, & ex prima in tertiam, fiunt 2 cubi p: 30 quadratis p: 72 positionibus æquantur 900, quare 1 cubus p: ... quadratis p: 36 positionibus æquantur 450.

... fficit igitur deducendo ad regulam, habere semper 1 cubum p: ... nero priorum quadratorum, addita ei quarta parte p: numero ...itionū tali, qualis est numerus æquationis primus, ut si habueris 1 q̃d q̃dratum p: 12 quadratis p: 36, æqualia 6 quadratis p, 60 ...sitionibus, habebimus 1 cubum p: 15 quadratis p: 36 positionibus æqualia 450, dimidio quadrati dimidij numeri positionum, & haberemus 1 q̃d q̃dratum p: 16 quadratis p: 64 æqualia 80 positionibus, haberemus 1 cubum p: 20 quadratis p: 94 positionibus æqualia 800, & si haberemus 1 q̃d q̃dratum p: 20 quadratis p: 100, æqualia 80 positionibus, haberemus 1 cubum p: 25 quadratis p: 209 positionibus æqualia 800, igitur hoc habito, in priore exemplo habemus, 1 cub. p: 15 quadratis p: 36 positionibus æqualia 450, igitur rei æstimatio, per 17^m capitulū, est ra: cubica 287½ p: ℞ 80449¼ ... ℞ v: cubica 287½, m, ℞ 80449¼ m: 5, hæc igitur est numerus quadratorum, qui duplicatus, est addendus ex utraq; parte, quia suppo... untur 2 res addendæ, & numerus adendus ex utraq; parte, ex demonstratione, est quadratum huius, cum eo quod fit ex hoc in 12, ...umerum quadratorum, manifestum est autem, quod ℞ quadrata ...rimi aggregati, semper est 1 quadratum p: dimidio numeri quadra...

Fig. 3. Here Cardan deals with biquadratics.

devoted solely to Algebra."[45] He uses p and m or p: and m: for plus and minus, and R for root. He does not seem to have used Paciolo's dash for an equality sign but simply expresses an equation by introducing the word aequale or aequantur. In the solution of equations he uses co., ce., cu., and the like in the same manner as Paciolo.

As appears from §9 Cardano used the same abbreviations co., ce., cu., —— as Paciolo, and applied them in the same manner in the solution of equations. The use of a notation for the second unknown is abruptly introduced in chap. li. of the Practica Arithmetica in the words, "Est & modus pulcher operandi per quantitatem surdam vt in exemplis." This introduces the following example in which co. and quan. are used, but not their product. There were three companions: the first said to the second and third: give me $\frac{1}{2}$ of what you have and I shall have 32; the second said to the first and third: give me $\frac{1}{3}$ of what you have and I shall have 28; and the third said to the first and the second: give me $\frac{1}{4}$ of what you have and I shall have 31. Cardano supposes that the second had "1 co." and the third "vnam quantitatem surdam," so that the first had "32 - $\frac{1}{2}$ co. - $\frac{1}{2}$ quan."; the second "28 - $10\frac{2}{3}$ + $\frac{1}{6}$ quan."; and the third "31 - 8 + $\frac{1}{8}$ quan. - $\frac{1}{8}$ co.": thus the second had "$17\frac{1}{3}$ + $\frac{1}{6}$ co. - $\frac{1}{6}$ quan." and the third had "23 - $\frac{1}{8}$ co. + $\frac{1}{8}$ quan."; and these quantities are equal to 1 co. and 1 quan. respectively. Thus $\frac{5}{6}$ co. is equal to $17\frac{1}{3}$ - $\frac{1}{6}$ quan. and $\frac{7}{8}$ quan. is equal to 23 - $\frac{1}{8}$ quan., so that $\frac{5}{6}$ co. + $\frac{1}{6}$ quan. is equal to 23. By subtraction we find that $\frac{17}{24}$ quan. - $\frac{17}{24}$ co. is equal to $5\frac{2}{3}$, so that 1 quan. - 1 co. is equal to 8.

Thus the second had 1 co., the third 1 co. + 8, and therefore the second and the third together had 2 co. + 8. Therefore half of what the second and third had is 1 co. + 4; whence it follows that the first had 32 - 1 co. - 4, that is, 28 - 1 co. The first and third therefore had 36: the third of this is 12, so that the second had 28 - 12 = 16, and therefore the first had 12 and the third 24.

This is the same question as that given by Paciolo in two examples which the companions have, after the transferences, are 96, 84, 81, and 50, 50, 50: and it is interesting to compare the methods adopted by Paciolo and Cardano.

Taking the equations to be (i) $x + \frac{1}{2}(y + z) = a$, (ii) $y + \frac{1}{3}(x + z) = b$, (iii) $z + \frac{1}{4}(x + y) = c$, Paciolo shows that $x + y + z = 2a - x$: and substituting $2a - x - y$ for $x + z$ in (ii) he finds that $y = \frac{3}{2}b - a + \frac{1}{2}x$; then substituting $2a - x - z$ for $x + y$ in (iii) he finds that $z = \frac{4}{3}c - \frac{2}{3}a + \frac{1}{3}x$: and substituting these values of y and z in (i) he obtains x. He takes x to be 1 co. and y and z are in turn 1 quantita.

Cardano deduces from (i) that $x = a - \frac{1}{2}y - \frac{1}{2}z$, and substituting this value of x in (ii) and (iii) he obtains the equations $\frac{5}{6}y = \frac{1}{6}z = b - \frac{1}{3}a$ and $\frac{1}{8} + \frac{7}{8}z = c - \frac{1}{4}a$, whence by subtraction $z - y = \frac{1}{17}(2a - 24b + 24c)$, = d, say, and therefore $z = y + d$, so that $\frac{1}{2}(y + z) = y + \frac{1}{2}d$, whence, from (i), $x = a - y + \frac{1}{2}d$, and therefore $x + z = a + \frac{1}{2}d$, whence from (ii) $y = b - \frac{1}{3}(a + \frac{1}{2}d)$. Cardano's 1 co. and 1 quantita are y and z.

The latter part of Cardano's solution is curiously inapt and cumbrous. When he has denoted y and z by 1 co. and quan. he follows a natural course in obtaining two equations connecting them, viz., $\frac{5}{6}y + \frac{1}{6}z = 17\frac{1}{3}$, $\frac{1}{8}y + \frac{7}{8}z = 23$. The direct procedure would then be to substitute for z from either of these equations in the other, e.g., the first equation gives $z = 104 - 5y$, whence substituting in the second equation we have $\frac{1}{8}y + \frac{7}{8}(104 - 5y) = 23$, giving $y = 16$. Instead of this Cardano subtracts one equation from the other, obtaining the equation $z - y = 8$. This is simpler than either of the relations between y and z given by the equations from which it is derived; but this simplicity is accidental. When he has obtained this relation $z - y = 8$, instead of substituting for z in either of the equations from which it was derived, as e.g., in the first, which gives $\frac{5}{6}y + \frac{1}{6}(y + 8) = 17\frac{1}{3}$, and so obtaining y at once, he finds the value of $\frac{1}{2}(y + z)$ which is $y + 4$, whence (i) he finds that $x = 28 - y$, so that $x + z = 28 - y + y + 8 = 36$; and therefore from (ii) $y + 12 = 28$.[46]

In the same chapter Cardan entitles a section "Regula de Medio" and explains that[47]

the form of this rule is to put $\frac{1}{2}$ quan. for the sum of the numbers sought, then divide $\frac{1}{2}$ quan. into two parts, of which one is 1 co. and therefore the other is $\frac{1}{2}$ quan.—1 co. It is here that he first gives the product 1 co. quan. He illustrates this by the problem: to find two numbers which multiplied together make 8, and whose squares joined to the numbers themselves make 27. He takes the sum of the numbers to be $\frac{1}{2}$ quan., so that the numbers are 1 co. and $\frac{1}{2}$ quan. - 1 co. The conditions of the question therefore give $\frac{1}{2}$ co. quan. - 1 ce. equal to 8. Thus $\frac{1}{2}$ co. quan. is equal to 1 ce. + 8.

[45] Smith, History of Mathematics, 1923, Vol. I, p. 297.
[46] Glaisher, Improvements, page 52.
[47] Ibid., p. 55.

Treating quan. as unity the values of 1 co. and $\frac{1}{2}$ quan. - 1 co. are (by rules for the solution of a quadratic equation) known to be $\frac{1}{4} + (\frac{1}{16} - 8)$ and $\frac{1}{4} - (\frac{1}{16} - 8)$.

The rules used by Paciolo and Cardan are derived from Euclid, which Cardan expresses as follows:[48]

1) Divide 24 into two parts such that 10 is a geometrical mean between them. The rule is equivalent to the formation of $12 \pm \sqrt{\{12^2 - 100\}}$, so that the parts are $12 + \sqrt{44}$ and $12 - \sqrt{44}$.

2) To find 1 co. from the equation 13 co. = 1 ce. + 30: the rule is equivalent to the formation of $\frac{13}{2} \pm \sqrt{\{(\frac{13}{2})^2 - 30\}}$ so that the values of 1 co. are $6\frac{1}{2} \pm \sqrt{12\frac{1}{4}}$, that is, they are 10 and 3.

3) If 10 is divided into two parts in extreme and mean proportion, the rule giving the parts is: square 10 giving 100, square its half giving 25, add these squares giving 125; then $\sqrt{125} - 5$ will be the larger part and $15 - \sqrt{125}$ the smaller part.

The last rule is of particular interest since a similar one was used almost exclusively by authors before the introduction of the modern formula for the solution of quadratics.

Glaisher points out the interesting fact that Cardan actually treats negative quantities in a problem, using the idea of debit and credit, thus giving a meaning to the solution, which would otherwise be considered impossible.

Both Paciolo and Cardan treat problems which were used generally in the subsequent period, namely those involving three numbers in continued proportion.

Although Glaisher gives a more comprehensive treatment of both Paciolo and Cardan, only such material as has direct bearing on the question under consideration is summarized here.

8. MICHAEL STIFEL:

Michael Stifel (1486-1567), called "the greatest German algebraist of the 16th century,"[49] was born in Esslingen and died in Jena. In 1544 he published the Arithmetica integra, in Latin. Its three parts treat respectively of rational numbers, irrational numbers, and algebra. The third part is entitled De numeris Cossicis, and de regula eorum, id est, de perfecta arte calculandi. In explaining the "Regula Algebrae," he writes on page 228:

Necque enim Algebrae regula curat denominationes uulgares, set columodo Cossicas denominationes curat: ut sunt hae denominationes, ℵ·3·Ⓒ·3ʒ·Ⓒ3·5·3Ⓒ· etc. Quae & numeros cossicos constituunt. Est enim numerus cossicus nihil aliud, quam numerus signo cossico denominatus.

In the next paragraph he explains that he calls the unknown the radix and not the cosa.

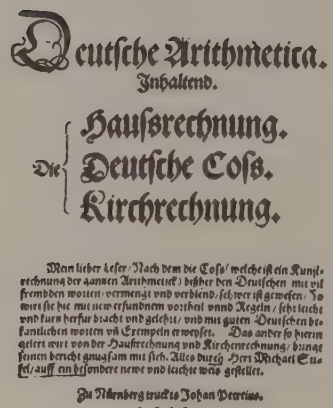
Under the heading "De secundis radicibus" Stifel discusses questions which involve more than one unknown, designating the first unknown by 1ℵ and the others, in order to avoid confusion, by 1A, 1B, 1C, up to seven unknowns. Since 1A, for example, stands for 1Aℵ, he then points out that 1A times 1A is 1Aʒ. In the appendix he includes the more interesting problems of Cardan, solving them, however, by the use of his own symbols, which he considers more convenient.

Among the examples cited by Stifel, Glaisher found problems which occurred in the works of Rudolff and Riese. The later authors used one unknown in the solutions, although Stifel employed 1ℵ and 1A. Stifel also resorted to Euclid for geometrical solutions of his problems. He also expressed the unknown by the use of the word "sum."

A citizen gave 7 denarioli to each of a certain number of poor men, and had 30 denarioli left: if he had given 9 denarioli to each he would have been short 30 denarioli. Here Stifel says that seven sums of denarioli, taken according to the number of poor men + 30 denarioli, is the total number of denarioli which he had. Thus 7 sum + 30 denarioli is equal to 9 sum - 30 denarioli, so that "1 summa makes 30", that is, the number of poor men.[50]

Stifel's Deutsche Arithmetica of which the Kuntsrechung or Deutsche Coss

[48]Glaisher, Improvements, p. 59.
[49]Cajori, A History of Mathematics, 1919, p. 140.
[50]Glaisher, Improvements, p. 86.

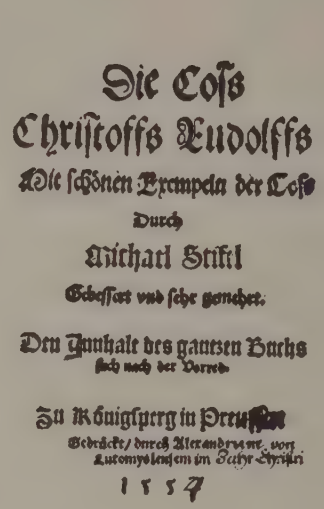


Deutsche Arithmetica.
Inhaltend.
Die Hausrechnung.
Deutsche Coss.
Kirchrechnung.

Mein lieber Leser/ Nach dem die Coss/ welche ist ein Kunstrechnung der gantzen Arithmetic) bisher den Deutschen mit vil frembden worten vermengt vnd verblendt gewesen/ also das sie bey mir new erfundnem vorzeit vnnd Regeln/ sehr leicht vnd kurtz herfur bracht vnd gesetzt/ vnd mit guten Deutschen deutlichen worten vñ Exempeln erwyset. Das ander so hierin gelert wirt von der Hausrechnung vnd Kirchenrechnung/ bringt seinen bericht genugsam mit sich. Alles durch Herr Michael Stifel/ auff ein besondere newe vnd leichte weis gestellet.

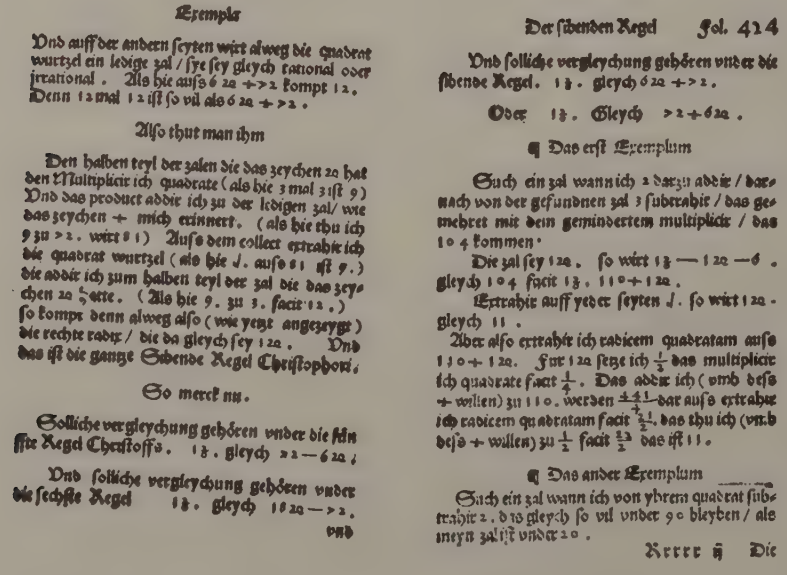
Zu Nürnberg truckts Johan Petreius.
1545.
Cum Priuilegio ad Quinquennium.

Fig. 4. Title page of Stifel, Deutsche arithmetica (Nürnberg, 1545).



Die Coss
Christoffs Rudolffs
Die schönen Exempeln der Coss
Durch
Michael Stifel
Gebessert vnd sehr gemehret.
Den Inhalt des gantzen Buchs
sich nach der Vorrede.
Zu Königsperg in Preussen
Gedruckt/ durchs Alexander Lutomysslensem im Jahr Christi
1554

Fig. 5. Title page of the 1554 revised edition of Rudolff's Coss by Stifel.



Exempla

Vnd auff der andern seyten wirt alweg die quadrat wurtzel ein ledige zal/ sye sey gleych rational oder irrational. Als hie aus 6 2a + 2 kompt 12. Denn 12 mal 12 ist so vil als 6 2a + 2.

Also thut man ihm

Den halben teyl der zalen die das zeychen 2a hat den Multiplicir ich quadrate (als hie 3 mal 3 ist 9) Vnd das product addir ich zu der ledigen zal/ wie das zeychen + mich erinnert. (als hie thu ich 9 zu 2. wirt 81) Aus dem collect extrahir ich die quadrat wurtzel (als hie √. aus 81 ist 9.) die addir ich zum halben teyl der zal die das zeychen 2a hatte. (Als hie 9. zu 3. facit 12.) so kompt denn alweg also (wie yetzt angezeygt) die rechte radix/ die da gleych sey 12a. Vnd das ist die gantze Sibende Regel Christophori.

So mercke nu.

Solliche vergleychung gehören vnder die fünffte Regel Christoffs. 1ß. gleych 2a — 6 2a.

Vnd solliche vergleychung gehören vnder die sechste Regel 1ß. gleych 10 2a — 2.

vnd

Der sibenden Regel fol. 414

Vnd solliche vergleychung gehören vnder die sibende Regel. 1ß. gleych 6 2a + 2.

Oder 1ß. Gleych 2 + 6 2a.

¶ Das erst Exemplum

Sach ein zal wann ich 2 darzu addire/ darnach von der gefundnen zal 3 subtrahir/ das gemehret mit dem geminderten multiplicir/ das 104 kommen.

Die zal sey 12a. so wirt 1ß — 12a — 6. gleych 104 facit 1ß. 11 0 + 12a.

Extrahir auff yeder seyten √. so wirt 12a. gleych 11.

Aber also extrahir ich radicem quadratam aus 11 0 + 12a. Für 12a setze ich ½ das multiplicir ich quadrate facit ¼. Das addir ich (vmb des + willen) zu 110. werden 440¼ dar aus extrahir ich radicem quadratam facit 21/2. das thu ich (vmb des + willen) zu ½ facit 22/2 das ist 11.

¶ Das ander Exemplum

Sach ein zal wann ich von yhrem quadrat subtrahir 2. das gleych so vil vnder 90 bleyben/ als mein zal ist vnder 20.

Rrrrr ij Die

Fig. 6. Two pages from the 1554 edition of Rudolff's Coss.

PRATICA
DARITHMETICA
DI
FRANCESCO GHALIGAI
FIORENTINO.

Nuouamente Riuiſta,& con ſomma
Diligenza Riſtampata.

IN FIRENZE
APPRESSO I GIVNTI
M. D. LII.

n°-----Numero ----1
ℓ°-----Coſa ----2
□ -----Cenſo ----4
⊠-----Cubo-----8
□ di □ -□ di □-----16
8 -----Relato---32
8 di □ -⊠ di □----64
⊞ -----Pronico---128
□ di □ di□-□ di □ di □--256
⊠ di ⊠---⊠ di ⊠ ----512
8 di □ ---8 di □---1024
⊞ -----Tronico -- 2048
⊠ di□di□-⊠ di □ di □-4096
⊞ -----Dromico--8192
⊞ di □ -- ⊞ di □--16384
⊠. 8 --- ⊠. 8 --- 32768

Fig. 7. Title page of Ghaligai's Practica d'arithmetica (1552), together with a part of fol. 72 which exhibits his designation of powers.

OPERE
DEL FAMOSISSIMO
NICOLO TARTAGLIA
CIOE'
Queſiti, Noua Scientia,
Trauagliata Inuentione, Ragionamenti ſopra Archimede,
NELLE QVALI COTIOSAMENTE SI SPIEGA,
L'Arte di Guerreggiare, coſi in Mare, come in Terra, col modo appreſſo di diffendere, offendere, & eſpugnare ogni gran Fortezza.
E Parte ancora del perfetto Bombardiero, & tutte le coſe à quello neceſſarie.
Si dimoſtra di più vn modo nuouo e mirabile di fabricare, & vſare, diuerſe ſorti di Squadre, & di Buſſole.
Et oltre maniere artificioſe, di cauare ogni gran V'aſello affondato, & fabricar due Vaſi con i quali ſi poſſa deſcendere nel fondo del Mare, & di ſua poterne ritornare di ſopra.
Et il Trattato de Inſidentibus aquæ, cioe delle materie che ſtanno, & che non ſtanno ſopra l'acqua, nel quale oltre la pratica ancor ſi vengono in queſto propoſito molti dubij conſiderationi
De locate all'Illuſtriſſimo Signor
LVIGI GIVSTINIANO
SIGNOR ALL'ARSENALE.

In Venetia, Al Segno del Lione. M. DC. VI.

Fig. 8. Title page of Tartagliа's Noua Scientia (Venice, 1606).

PARVA RES CRESCVNT.
ALGEBRA
APPLICATA
Douè ſi moſtra la vtiliſsima applicatione d'eſſa alla inuentione delle coſe recondite nelle diuerſe Scienze, & Arti.
DI PIETROANTONIO CATALDI LETTORE DELLE SCIENZE Mathematiche nello Studio di Bologna.
ALL'ILLVSTRISS. SENATO
DI BOLOGNA

IN BOLOGNA, M. DC. XXII
Per Nicolò Tebaldini.
Con Licenza de' Superiori.

Fig. 9. Title page of Cataldi's Algebra Applicata (Bologna, 1622).

was a portion, was published in 1545. The author's aim in this part was to substitute cossic methods for the rule of false, but without using cossic signs or Latin words. He points out that "the cossic method is far preferable to the rule of false, and that it is the epitome of all the ingenious rules of arithmetic, and compared to it, the rule of falso is but as a point to a circle."[51]

Stifel begins his Deutsche Coss by explaining that he uses the ordinary numbers 1, 2, 3, ... and that these numbers are also used under the summa.

And this name summa is also denoted by sum̄ as 1 sum̄: 2 sum̄: 3 sum̄: 4 sum̄: etc. Only do not think that you are learning something peculiar or different. For summa here means nothing different from the word as it is used in common German speech, as when we say of a rich man: he has put a great sum of money or of florins, etc. And I may put 1 sum̄ for a number of florins or for a number of denarioli or grossi, or men or houses or whatever I will. So 1 sum̄ fl. denotes to me a heap of florins, for I may not yet have counted or reckoned them. And we may (according to this) call unreckoned numbers 1 sum̄: 2 sum̄: ...[52]

He goes on to explain that 3 times 4 sum: makes 12 sum: and that 2 sum: by 3 sum: is 6 sum: sum: He gives the latter multiplication in the following:

But if I multiply 2 sum: by 3 sum: I get 6 sum: sum: This I may also read as 6 summa summarum, as one often finds in German summa summarum. Or if you wish to have it still in German and more intelligible I can call it 6 sums multiplied by 1 sum.[53]

It is of interest to note his manner of designating second unknowns.

Thus, if there are such unreckoned numbers of three kinds, one may remain unspecified, as you see here, 1 sum:, 1 sum: A, 1 sum: B, etc.[54]

His quadratic equations are expressed as 1 sum: sum: equate to some such expression as 112 - 6 sum:

He concludes his Deutsche Coss by indicating that the radices secundae can be studied in another way; that is, by writing the second power of 1A as 1AA, and the like.

GLAISHER comments that "from an algebraic point of view his replacement of a sign by a word for the unknown in the Deutsche Coss was a step backward. But the changes he made were forced upon him by the motive of the book, which was to show that algebraic methods were preferable to the rule of false, and that the cossic signs and Latin words were not essential in the simpler applications of algebra."[55]

Nine years after the publication of the Deutsche Coss, appeared STIFEL'S edition of Rudolff's Coss, 1553. After giving all of RUDOLFF'S examples, he adds an appendix containing 24 examples of his own.

9. TARTAGLIA:

Nicolo Brescia (1490?-1557) or Tartaglia, (the Stammerer) is the author of Il General Trattato di Numeri e Misuri, 1556-1560, six parts in three volumes infol., the last part of which treats of algebra.

In La Noua Scientia de Nicolo Tartaglia con una gionta al terzo Libro, Venetia, 1637, appear the symbols for the successive powers as n, co, ce, cu, cece, pri., rel., and similar ones. In the section, fol. 98-128, he includes questions as they were proposed by different authors. For example, he gives (fol. 99'),

Questo Ottavo Fatto da Maestro Francesco Feliciano l'anno. 1526. in Verona.

Maestro Francesco. Vno compra una quantita di pernice, & spende in tutte lire. 25. de denari, & le prota a Mantuca a reuendere, & tante, come che lui ne haue all lire, lui li uendete tante lire l'una, & le uendete tutte, accetto che. 10. & ne cauo lire .75. de denari.

This is followed by the solution of the problem, in which he obtains the equation

[51] Glaisher, Improvements, p. 87.
[52] Ibid., p. 93.
[53] Ibid., p. 94.
[54] Ibid., p. 96.
[55] Ibid., p. 99.

25. censi men. 10. co. le quale saranno equale a' lire .75.

(In modern notation: $25x^2 - 10x = 75$.) Dividing the equation by 25, results in

$$x^2 - \tfrac{2}{5}x = 3.$$

He goes on to say that he obtains the final solution by "the following rule," which is not cited, although the value of cosa is given as

$$x = \sqrt{.3.\ \tfrac{1}{25} + \tfrac{1}{5}} \text{ or } \sqrt{3 + \tfrac{1}{25}} + \tfrac{1}{5}.$$

Tartaglia seemed to avoid the other root, but he was most assuredly unafraid of irrational roots.

The authors of this period frequently mentioned the "impossible" solution, and Tartaglia was no exception. On fol. 102 he quotes a problem of Alessandro Venetiano, 1533, in Verona:

Maestro Alessandro. Essendo io in Fiorenza gia sa quatro mesi & mi fu da ta una ratione la qual son certo ch'eglie impossible a rissoluerla, come credo che il medesimo uoi affermareti.

The problem is then stated in the form:

Given a triangle of three unequal sides, the base is equal to 10, the perpendicular to 8 and the other two together are 20. It is required to find the value of each of the other two sides.

After setting up the problem in his usual rhetorical manner, he finally arrives at the solution, with the roots as,

$$.10.\ \text{men. R } 3.\tfrac{2}{3}, \text{ or } (10 - \sqrt{3\tfrac{2}{3}})$$

$$.10.\ \text{piu. R } 3.\tfrac{2}{3}, \text{ or } (10 + \sqrt{3\tfrac{2}{3}}).$$

Here TARTAGLIA did not avoid the two roots, as seemed apparent in the first problem quoted above, but he must have considered both, since they were needed for a complete solution.

Geometrical solutions were also employed.

10. GIELIS VAN DER HOECKE:

An early Dutch algebra was published by Gielis van der Hoecke, which appeared under the title, In arithmetica een sonderlinge excellent boeck (1537), in which he gives fourteen rules for the solution of equations. These include equations which may have negative as well as positive terms. The fourth rule[56] which adequately illustrates the general method explained in the other rules, deals with the equation of the type $ax^2 + bx = N$, or as the author says,

the two greater are equal to the lesser or the lesser equal to the two greater.

He proceeds to direct the division of the two lesser terms, meaning, of course, the two terms of lower degree by the coefficient of the greater or highest degree term. Then $\frac{1}{2}$ of the coefficient of the middle term is taken, squared, and added to the least term. From the square root of this sum $\frac{1}{2}$ of the middle term is subtracted, giving the value of the unknown. The example given is $2x^2 + 3x = 14$. Van der Hoecke divides through by 2, as directed above, giving $x^2 + 1\frac{1}{2}x = 7$. $\frac{1}{2}$ of $1\frac{1}{2}$ is $\frac{3}{4}$ which squared is $\frac{9}{16}$. This added to 7 gives $7\frac{9}{16}$, the square root of which is $2\frac{3}{4}$. $\frac{3}{4}$ which is $\frac{1}{2}$ of the coefficient of the middle term subtracted from $2\frac{3}{4}$ leaves 2 as the value of x.

On page 84^{v} he explains his manner of treating fractional terms in the example,

$$\frac{2 \text{ pri.}}{3 \text{ nu.}} = \frac{6 \text{ nu.}}{1 \text{ pri.}} \qquad \frac{2x}{3} = \frac{6}{x}.$$

He cross multiplies, "duceert enn multipliceert int cruyce," with the result as

18 N dit is ghelijck 2 se.

or

$$18 = x^2$$

which falls under Van der Hoecke's second rule for equations.

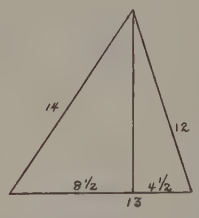
[56] AEgidius van der Hoecke, Een sonderlinge boeck dye edel conste Arithmetica, 1537, p. 79^{v}.

We will cite two examples that are typical of the applications van der Hoecke made of the rules of algebra. The first[57] one deals with a man who possessed a certain number of pennies. Upon going from Ghent to Antwerp the amount was immediately doubled of which, after spending 10 pennies, he still had 6 pennies left. Van der Hoecke assigns 1 pri (x) for the number of pennies he originally had. According to the conditions of the problem he formulates the equation

$$2 \text{ pri} - 10 = 6 \qquad 2x - 10 = 6$$

giving 8 as the value of x or the number of pennies he originally had.

The other problem[58] is concerned with a triangle with three unequal sides, of which the smallest is 12 feet, the largest 14 feet and the base is 13 feet. In order to know on what point the perpendicular should come to rest you must know the angle in the base which should make two orthogonal triangles. I take 1 pri (x) for the first part of base; then I will cut off from the 13-foot line 1 pri and multiply the smallest side into itself, as 12; which becomes 144. Then multiplying the smallest base into itself, it will become 144...



In this same manner van der Hoecke arrives at the equation:

$$196 - (13 - x)^2 = 144 - x^2$$

whose solution gives $4\frac{1}{2}$ for one segment and $8\frac{1}{2}$ for the other.

[57] AEgidius van der Hoecke, Een sonderlinge boeck dye edel conste Arithmetica, 1537, p. 77^{v}.
[58] Ibid., p. 90^{v}.

Chapter IV

LINEAR AND QUADRATIC EQUATIONS 1550-1660

NESSELMANN, in his Die Algebra der Griechen[59] divided the development of algebra into three periods, known as Rhetorical Algebra or "reckoning with words"; Syncopated Algebra, chiefly rhetorical, interspersed with a few symbols; and finally, Symbolic Algebra, which title is self-explanatory. Modern mathematics belongs to this latter stage. This division is largely fanciful so far as historical development is concerned, but is often cited as though it corresponds to historical realities. All through the ages from the time of DIOPHANTUS and even with the Babylonians traces of all three stages appeared together. As late as Gemma Frisius, the little algebra that is found in his Arithmeticae is treated in an entirely rhetorical manner, as for example (p.Iiiij):

> Vocamus autem quadrati quadratum numerum, qui eductu quadrati alicuius in se ipsum producitur, vt 9 cum sint quadratum de 3, erunt 81 quadrati quadratum, et ratione hac 3, radicis radix de 81, radix enim de 81, valet 9, huius item radix 3.

BALL, in his History of Mathematics,[60] used this classification, placing Chuquet, Paciolo, Recorde, Rudolff, Riese, Stifel, Tartaglia, Cardan, Ramus, Xylander, Peletier, Romanus, and Bombelli as representatives of the syncopated stage. Neither Peletier nor Bombelli, for example, belongs in this class. As we shall see later, both of them, Bombelli particularly, used a very complete system of symbols, although they never became standardized.

Furthermore, Ball's statement that

> It is evident that Stifel and Bombelli and other writers of the 16th century had introduced or were about to introduce some ideas of symbolic algebra.[61]

is too restrictive, as Glaisher and Bortolotti have shown in regard to Stifel and Bombelli, respectively.

It is hoped here to give a comprehensive view of the progress of the fundamental principles governing simple and quadratic equations made in the period 1550-1660. The material discussed will be dealt with under the following topics:

1. Definition, Formulation, and Classification of Equations
2. Symbolism and Terminology
3. Reduction of Equations
   a) Simple Linear Equations
   b) Simultaneous Equations
   c) Quadratic Equations
4. Geometrical Demonstrations
5. Typical Problems

1. Definition, Formulation, and Classification of Equations

Algebra was essentially the study of the equation for most of the authors. PEREZ DE MOYA (1573) states in his work that[62]

> Algebra is a way of finding some unknown desired number subjected to some proportionality, for which it is designated also by the name of calculation made by the progression of geometric proportionality. It is called by different names; it is called the rule of algebra by some people, which means restauration. It is called by another Almucabula, which in Arabic means position. It is called by another the rule of Cosa. All names of effect, for acting on it (with the idea of looking for some number which possesses some property or properties) we assume the number we look for to represent some object or something of this kind. And due to this

[59] H. G. F. Nesselmann, Die Algebra der Griechen, Berlin, 1842, p. 302.
[60] W. W. Rouse Ball, A Short Account of the History of Mathematics, 1919, pp. 201-228.
[61] Ibid., p. 229.
[62] Trattado de Mathematicas, 1573, p. 429--Moya, Trattado.

position it is called differently in different languages, and from it results the diversity of its various names. Buteon calls it, with no reason, quadrature, because it treats not only squares but cubes and other roots. Nobody used this name. Let it be named at the pleasure of everyone, for its end is the same; that is to show how to find some proportional number or numbers (ignored and demanded if it is possible to be given, and to notify us of questions which are possible), taking for this purpose as a medium the proportion of equality.

DIGGES (1594) says more concisely,[63]

... Of these AEquations I will speake the more particularly, for all the operations of Algebra tend to this finall end, to frame an AEquation, and then thereby to search the values of the Roote, or Prime, whereby the most difficult questions that may arise or be propounded, are with farre more facilitie to be resolued, than by any other Rules whatsoeuer.

In some cases there are but a few pages devoted to the fundamental operations, as in Perez de Moya; and in others, Gemma Frisius for example, the order of their consideration is reversed. Many, however, were impartial and assigned proportionate space to each division of the study of algebra. In order to get a systematic idea of the development of the equation in the period in question, we shall begin our study by making a comparison of the definitions of the equation as given by some of the authors. Not all of them attempted such a definition; some apparently assumed some previous knowledge or left it to the ingenuity of the teacher to supply such knowledge.

Although GEMMA FRISIUS (1550) gives very little on the equation, he does define it as,[64]

Proportio aequalitatis est, dum duae quantitates aequales ad inuicem comparantur.

We translate the definition of SCHEUBEL (1552):[65]

An equation, as it is taken here, just as the name indicates, is an expression in which two things are shown to be equal to each other.

We find in BOMBELLI (1579) that[66]

L'Agguagliare non è altro, che hauere due quantità, ò semplici, ò composte sotto diuersi nomi, le quali due quantità ancor che siano di diuersi nomi, nondimeno uagliono egualmente, però se si lieua ad una, bisogna leuare quel medesimo all' altra, e se si aggionge ad una il medesimo si aggionge all'altra, e quesito si sà per la infallibile propositione.

Salignacus (1580) tells us that the algebraic equation which explains arithmetical questions is to[67] "equate heterogeneous figures by certain laws."

Digges (1594) passes from the study of the fraction to that of the equation by the transitional phrase, "Now to the Equation," beginning this section by defining it thus:[68]

Aequation is nothing else but certaine conference of two numbers being in value equal, and yet in multitude and Denomination different.

Graffenried (1619) quotes P. Ramus:[69]

Theilet sie inn zwen (sic) Theil ab/als in Numerationem und AEquationem, das ist/inn den Algorythmum der Cossischen Zahlen und Zeichen und in die AEquation, oder vergleichung zwener (sic) dingen/und ist diese Kunst also gegründet inn Acht Regeln der AEquation.

Follinus (1622) briefly states that[70]

AEquatio hic est proportio aequalitatis inter duas quantitates siue res varie denominatas.

Moore (1660) gives exactly the same definition as Follinus; while Vav-Lezard (1630) and Hume (1636), both editors

[63] Leonard and Thomas Digges, Stratioticos, 1594, p. 44.
[64] Gemma Frisius, Arithmeticae practicae Methodus facilis, I VII.
[65] J. Scheubel, Algebrae Compendiosa, 1551, fol. p. 18^{v},--Scheubel, Algebrae.
[66] L'Algebra Opera, p. 238--Bombelli, Algebra.
[67] Arithmeticae, p. 132.
[68] Stratioticos, p. 44.
[69] Arithmeticae Logisticae, p. 230--Graffenried, Logisticae.
[70] Algebra, p. 180.

of Vieta's work, say that the equation is a comparison of a known quantity with an unknown quantity. This is essentially the definition of OUGHTRED (1647):[71]

----its terms are so ordered, that those things which are given in measure, may make one part of the AEquation; and those which being unknown, are sought, the other.

In 1639 STAMPIOEN said that[72]

Die dinghen, die elcx (sic) ghelijck zijn een ander dingh, die zijn malckanderen ghelijck.

GIBSON (1655) is the only one besides Recorde who, in defining an equation, indicates the use of the sign of equality:[73]

An AEquation is when one or more special quantities are equal to one or more other special quantities, and written with the signe of equality betwixt them.

RENALDINI gives a definition similar to that of Follinus with an addition from Vieta:[74]

Aequalitatis proportio inter duas quantitates, vel res varie denominatas, seu certam notam ex una parte, ex alia incertam, & ignotam, & ut loquitur Vieta, est magnitudinis incertae cum certa comparatio.

The following authors, Feliciano (1550), Juan Diez (1556), Visconti (1581), Unicorno (1598), Petri (1605), Ceulen (1615), Cataldi (1619), Girard (1629), De Billy (1637), Gloriosi (1639), Lauremberg (1643), Grisio (1644), Herigon (1644), Peucer (1656), Foster (1659), and Rahn (1659), give no formal definition of an equation, but introduce the subject by a rule for its formation.

Some seem to assume these fundamentals and take up directly the discussion of the reduction of an equation. For example, RECORDE (1557) explains:[75]

When any question is propounded, appertainyng to this rule, you should imagine a name for the number, that is to bee soughte, as you remember, that you learned in the rule of false position. And with that number shall you procede, accordying to the question, until you finde a Cossick number, equalle to that number, that the question expresseth, which you shall reduce evermore to the leaste nombers, etc.

ANTICH ROCHA treats the equation by specifying "convenient rules to understand better the equations ... and this is the first rule." He then tells us to call the thing demanded, cosa and

you will imagine that the thing asked or demanded is already done or answered, but you have to prove it. You will write first that the answer is cosa, with which you will go on and work as it is asked, and what you get with the first thing is what you want it to be;...

In the original the statement reads as follows:[76]

Digo que para hazer qualquier demanda por esta dela Cossa, has de imaginar que la tal demanda es ya hecha y respondida, empero tu la quieres prouar; y pornasprimeramente que la respuesta fuesse vna cosa, con la qual has de proceder haziendo lo que la demanda pidiere,...

The interesting fact about the subsequent rules as well as the problems is that Rocha used verbatim the statements of Marco Aurel (1552). CLAVIUS (1608) expresses the "Regula Algebrae" thus:[77]

Pro numero incognito quaestione ponatur radix una hoc modo. 1$\mathcal{L}$ ... Quae iuxta quaestionis tenorem examinetur, donec aequatio aliqua inueniatur:...

Henrion (1615), using 1$\mathcal{R}$ instead of 1$\mathcal{L}$, and Hume (1635), designating the unknown by i, duplicate Clavius' statement in French; Follinus (1622), and De Billy (1637) repeat it. PEUCER completes his rule by[78]

[71] The Key of the Mathematicks, p. 54--Oughtred, Key.
[72] Algebra, p. 35.
[73] Syntaxis Mathematica, p. 25, Gibson, Syntaxis.
[74] Opus Mathematicum, 1644, p. 122, Renaldini, Opus.
[75] The Whetstone of Witte, p. Ge IV^{r}.
[76] Arithmetica, 1565, p. 262.
[77] Algebra, p. 36.
[78] Valentini Nabodi de Calculatoria Numerorumque natura Sectiones quatuor, p. S VI^{r},----Peucer, Sectiones.

Si propositum fuerit vel aenigma aliquod quod sit explicandum, vel aliud quidpiam, quod sit investigandum numeris ac proferendum: assume pro eo 1℣. Et iuxta tenorem questionis quae aenigmatis circumstantias enumerat ac declarat, positam 1℣ auge vel minue, quoad perueneris ad aequationem inter duos numeros diversimode repraesentatos.

SESEN (1617) and GRAFFENRIED (1619) give a rule that is almost identical to that of Peucer. The latter says,[79]

Und mit solchem gesetzten Radix, muss man darnach procedieren in aller gestalt sampt wer es die erste Zahl so lang biss die sach dahin gebracht dass zwo Ordnung der Zahlen einandern gleich werden.

Let us observe the similarity of OUGHTRED'S rule to that of Aurel and Rocha.[80]

Whensoever any Problem or Question is propounded, suppose that done which is desired; and having used a fit Rationation for the Magnitude sought, put A or some vowel; and for the Magnitudes given, put Constants; that thereby the Magnitudes given, may easily be distinguished from those that are uncertain.

MOORE (1660) also uses Oughtred's idea but couples it with that of Descartes, saying that[81]

When a question in Arithmetik is asked you may place for the quantity or number sought A, or any vowel, expressing those things which are known by Consonants and the unknown or which is sought vowels; but after Des-Charles if you practice with small letters, then mark all the given quantities with the beginning of the alphabet a, b, c, d, etc. and the unknowns by the last letters x, y, z, etc. It must be examined (as though you had it given, were proving the truth of it) by some of these means following, till you can bring it to an AEquation.

VAN DER HUIPS, whose edition of 1661 was the one examined, since it is not likely that it is essentially different from the edition of 1654, shows marked influence of Stampioen and Descartes.[82]

In plaatze von het onbekende, 't welck offbegeert wert ofte eenen toe-ganck en wech tot 't begeerde is, stelt altoos eenighteecken als 1x, 1 ① ofte, &c naar den en over-een-koomingh van 't gegevene, oock zo-danigh: dat 't zelve de gemeene Tellinghon-onder-worpen is.

Where any classification of equations occurs, it usually follows the definition of the equation and, in general, divides them into simple and composite equations. Representative authors are Rocha (1565), Perez (1573), Visconti (1581), Herigon (1634), De Billy (1643), and Renaldini (1644). Gibson speaks of the simple and mixed equations. Clavius shows the direct influence of Leonard of Pisa by giving the same division into "Tres simples, tres, compositae." Herigon adds affected quadratic, biquadratic and cubic equations. Vieta speaks of the pure and affected quadratic equations. GIRARD specifies four classes:[83]

I. DEFINITION

Equation simple, est celle qui n'a qu'une quantité esgale à un nombre: autrement elle est dite Composée ou Meslée.

II. DEFINITION

Equation complette, est celle qui a toutes les quantitez sans en delaisser pas une.

III. DEFINITION

Equation primitive est quand les denominateurs des quantitez sont eutr'eux premiers.

IV. DEFINITION

Equation derivative est quand les denominateurs des quantitez sont entr'eux composez.

He also states on page 24 that equations are homogeneous when (1) is equated with (1), that is x is equal to x and heterogeneous when (0) is equated to (1), that is a constant is equal to x.

ALSTED (1641), in his Method Admirandorum Mathematicorum, which De Morgan qualifies as "a slight treatise on arithmetic with a few words about algebra,"

[79] Logisticae, p. 230.
[80] Key, 1647, p. 53.
[81] Arithmetick, p. 54.
[82] Algebra, p. 112.
[83] Invention nouvelle en l'Algebre, p. 38, Girard, Invention.

left us an interesting classification of equations.

SALIGNACUS (1580) gives his classification into prima aequatio as[84]

Aequatio Algebraica duplex est. Prima duobus inter se aequatis, tertium absoluto aequalem per proportionem concludit.

and

Aequatio secunda est tribus heterogeneis notarum continué proportionalium duos aequat uni, ut quadrato dimidio numeri valorem unius medii inveniat.

This second class he subdivides into primus generis and secundi generis, which will lead us into the discussion of the general types of equation, a heritage of Al-Khowarismi. In giving examples of the primus generis, Salignacus, of course, puts them into his own notation.

1q + 8ℓ & 65 $x^2 + 8x = 65 (ax^2 + bx = N)$

6ℓ + 40 & 1q $6x + 40 = x^2 (ax^2 = bx + N)$.

For the secundi generis he uses

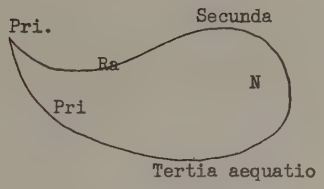
1q + 2 & 10ℓ $x^2 + 2 = 10x (ax^2 + N = bx)$.

This division is strikingly similar to that of Scheubel.[85]

1. Prima aequatio est, ubi duae quantites vel duo numeri, diversis characteribus signati, inter se aequales esse proferuntur.

8 radices	16N	$8x = 16$
9 primae	18 ra quot	$9x^2 = 18x$
unitatibus aequatur 2.		$x = 2$
6 secundae aequatur		$6x^3 = 24x^2$
24 pri. una radix facit 4.		$x = 4$, etc.

2. Secunda aequatio est, ubi tres numeri tribus diversis, continuatis tamen, characteribus signati, duo uni, vel unus item duobus aequalis esse profertur.



[86]

a. Primus canon
Pri. ra. aequales N
b. Secundus canon
Ra. N aequales pri.
c. Tertius canon
Pri. N aequales ra.

RECORDE (1557), who was unquestionably influenced by Scheubel, explains his digression from his predecessor's classification of types of equations. He says that[87]

This rule of equation is divided by some men into diverse partes. As namely, Scheubelius make .3. rules of it. And in the second rule he dooeth putteth .3. severalle canons. Some other men make a greater nomber of distinctions in this rule. But I intende------to distinct it in two parts. Wherefore the first is, when one nomber is equalle unto one other. And the seconde is, when one nomber is compared as equal unto .2. other nombers.

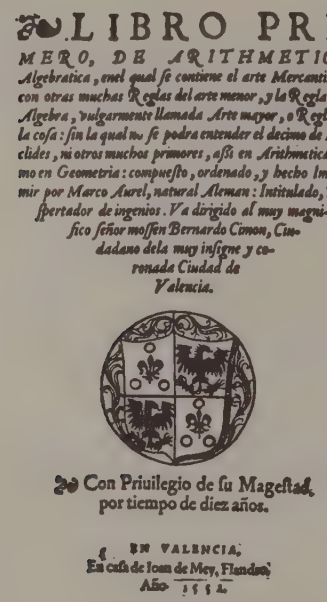
He often makes the self-crediting statement concerning these classes, as for example, when he speaks of[88]

The second kinde of equation, after Scheubelius minde and myne also, is, when one simple nomber Cossike, is compared as equalle to .2. other simple nombers Cossike, of severalle denominations, and like distaunce.

Each time he seems to wish to impress the fact that although he is dependent on Scheubel, he still lays some claim to originality. This is upheld by the statement on the next page:

But now bicause Scheubelius dooeth make .2. severalle equations diverse rules, or canons for each of them, I will declare his .6. canons to be all contained in this seconde kind of equation.

[84] Arithmeticae, pp.134, 138.
[85] Algebra, p. 12^r.
[86] Ibid., p. 18^r.
[87] Whetstone of Witte, p. Ff. iv.
[88] Ibid., p. Ff. $iiij^r$.

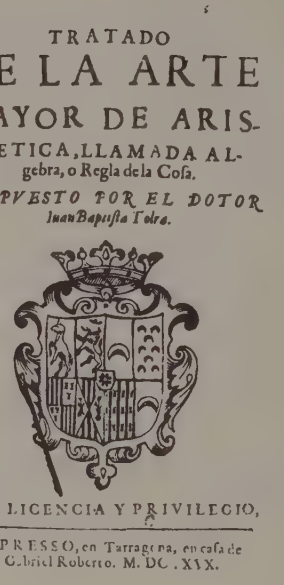


LIBRO PRIMERO, DE ARITHMETICA Algebratica, enel qual se contiene el arte Mercantiuol, con otras muchas Reglas del arte menor, y la Regla del Algebra, vulgarmente llamada Arte mayor, o Regla de la cosa: sin la qual no se podra entender el decimo de Euclides, ni otros muchos primores, assi en Arithmetica como en Geometria: compuesto, ordenado, y hecho Imprimir por Marco Aurel, natural Aleman: Intitulado, Deo spertador de ingenios. Va dirigido al muy magnifico señor mossen Bernardo Cimon, Ciudadano dela muy insigne y coronada Ciudad de Valencia.

Con Priuilegio de su Magestad, por tiempo de diez años.

EN VALENCIA, En casa de Ioan de Mey, Flandro. Año 1552.

Fig. 10. Title page of Aurel, Arithmetica Algebratica (Valencia, 1552).

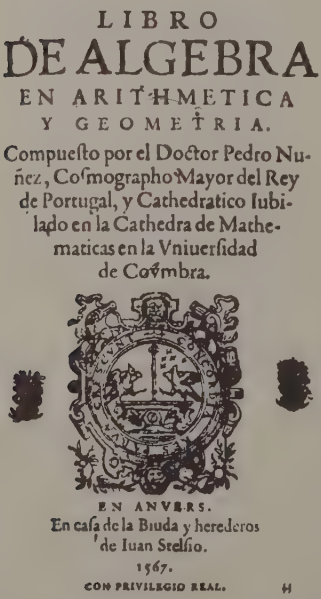


TRATADO DE LA ARTE MAYOR DE ARISMETICA, LLAMADA Algebra, o Regla de la Cosa. COMPVESTO POR EL DOTOR Iuan Baptista Tolra.

CON LICENCIA Y PRIVILEGIO,

IMPRESSO, en Tarragona, en casa de Gabriel Roberto. M. DC. XIX.

Fig. 11. Title page of Tolra, Tratado de la Arte Mayor de Arismetica, Llamada Algebra, o Regla de la Cosa (Tarragona, 1619).



LIBRO DE ALGEBRA EN ARITHMETICA Y GEOMETRIA. Compuesto por el Doctor Pedro Nuñez, Cosmographo Mayor del Rey de Portugal, y Cathedratico Iubilado en la Cathedra de Mathematicas en la Vniuersidad de Coymbra.

EN ANVERS. En casa de la Biuda y herederos de Iuan Stelsio. 1567. CON PRIVILEGIO REAL.

Fig. 12. Title page of Nuñez, Libro de Algebra en Arithmetica y Geometria (Anvers, 1567).

LA ARISMETICA DE IVAN VENTALLOL, TRADVZIDA DE LENGVA Catalana en Castellana, por el Doctor Juan Batista Tolra.

Va añadido un Tratado de la ARTE MAYOR llamada ALGEBRA o REGLA DE LA COSA. Compuesto por el mismo Doctor Tolra.

Dirigido al Ilustrissimo Señor D. IVAN de BOXADOS, y de PAX Conde de Savellà, Señor de la Baronia de Vallmoll, y gentilhombre de la boca del Rey nuestro Señor.

Año 1619

CON LICENCIA Y PRIVILEGIO.

En Tarragona, por Gabriel Roberto.

Fig. 13. Title page of John Ventallol, La Arismetica translated from Catalan into Castilian by John Tolra (Tarragona, 1619).

ANTICH ROCHA (1565) in introducing his classification summarizes the number of equations employed by some of the earlier writers.[89]

Fra Lucas de Burgo puso .6. ygualaciones Marco Aurel Aleman puso .8. Albertucio de Saxonia puso .10. Esteuan de la Roche puso .4. Ioan Scheubelio .3. Perez de Moya .7. aunque ala verdad pone .8. y despues las reduze a .4. he determinando de seguir a Marco Aurel Aleman, y traerte .8. ygualaciones, en las quales estan fundadas las respuestas desta regla: las .4. son simples, y las .4. son compuestos.

Translating this freely we understand that

Brother Lucas de Burgo wrote 6 equations, Marco Aurel wrote 8, Albert of Saxony wrote 10, Estienne dela Roche wrote 4, John Scheubel 3, Perez de Moya 7, although he really means 8, and later he reduces them to 4. I have decided to follow Marco Aurel in his 8 equations in which are involved the answers to this rule. 4 of them are simple and 4 compound.

AUREL (1552) himself says that he is giving eight equations, although Brother Lucas of Burgo gave 6, and Albert of Saxony, 10. He takes the arithmetic mean of 6 and 10 which is 8 and treats of four equations which are called simple and are composed of two quantities; whereas the other four equations are called composite, and are composed of three quantities, as will be demonstrated.[90]

1. $ax^3 = bx^2$
2. $ax^2 = N$
3. $ax^4 = bx$
4. $ax^4 = N$
5. $ax^2 + bx = N$
6. $ax^2 + N = bx$
7. $ax^2 = bx + N$
8. $ax^4 + bx^2 = N$
   $ax^6 + N = bx^3$
   $bx^4 + N = ax^8$

VISCONTI (1646) on page 129 of his Practica Numerorum & Mensuram tells "De tribus compositis canonibus":

Viso canone simplici, reliquum est, vt tres compositos inspiciamus. Qui compositi dicuntur, quia post aequationem factam, semper in continente remanent duae quantitates diuersae, & in contento vna sola, ex qua ipsi canones inter se dignoscuntur. Nec plures habentur, quia in eis de tribus tantum quantitatibus agitur, que sunt numerus agitur, linea & quadratum nam cubum prorsus non recipiunt.

In primo canone numerus est semper contentum.
In secundo quadratum.
In tertia linea.

This is exactly the same classification as that of Buteo. Here we find the word linear applied for the first time to any equation.

STAMPIOEN (1546) tabulates his classification of equations in the following manner:[91]

+1 ② + ①		⓪	Eerste
1 ② − ①	ghelijck	⓪	Tweede
1 ② − ①		−⓪	Derde voor-vall

and RENALDINI (1655):[92]

Q − R = N	2	1	0
Q + R = N	2	1	0
R − Q = N	1	2	0

In each case we see that if the negative term is transposed the equation will assume the form given by Al-Khowarismi. The Arabic numerals following Renaldini's equations simply indicate the position of the constant and of the first and second power terms.

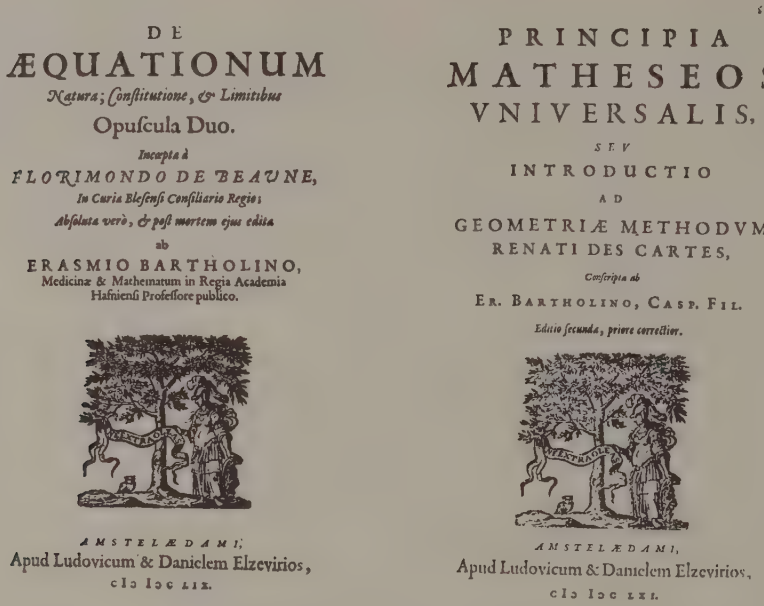
MOORE (1660) indicates his three types as,[93]

ZA − Aq = AE
Aq − XA = AE
Eq + XE = AC

while FLORIMOND DE BEAUNE in Descartes' geometry (1661) comes nearer to the modern notation:[94]

$xx + lx + mn \propto 0$
$xx - lx + mn \propto 0$
$xx - lx - mn \propto 0$

[89] Arithmetica, p. 264.
[90] Libro Primero, De Arithmetica Algebratica, Fol. 77r----Aurel, Arithmetica.
[91] Algebra, p. 44.
[92] Opus, p. 125.
[93] Arithmetick, p. 69. [94] De Natvra Aeqvationvm in Geometriae, 1661, p. 64.



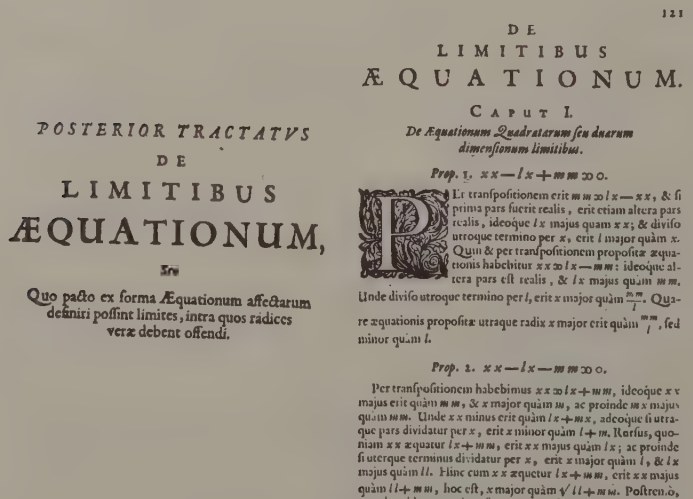
DE
ÆQUATIONUM
Natura; Conſtitutione, & Limitibus
Opuſcula Duo.
Inceptа à
FLORIMONDO DE BEAUNE,
In Curia Bleſenſi Conſiliario Regio:
Abſoluta verò, & poſt mortem ejus edita
ab
ERASMIO BARTHOLINO,
Medicinæ & Mathematum in Regia Academia
Hafnienſi Profeſſore publico.

AMSTELÆDAMI,
Apud Ludovicum & Danielem Elzevirios,
CIↃ IↃC LIX.

PRINCIPIA
MATHESEOS
VNIVERSALIS,
SEV
INTRODUCTIO
AD
GEOMETRIÆ METHODVM
RENATI DES CARTES,
Conſcripta ab
ER. BARTHOLINO, CASP. FIL.
Editio ſecunda, priore correctior.

AMSTELÆDAMI,
Apud Ludovicum & Danielem Elzevirios,
CIↃ IↃC LXI.

Fig. 14. Title page of the 1659 edition of De AEquationum Natura by Florimondo de Beaune. It is noteworthy that this and the following text were edited by Erasmus Bartholinus.



POSTERIOR TRACTATVS
DE
LIMITIBUS
ÆQUATIONUM,
Seu
Quo pacto ex forma Æquationum affectarum definiri poſſint limites, intra quos radices veræ debent offendi.

DE

121

DE
LIMITIBUS
ÆQUATIONUM.

CAPUT I.
De Æquationum Quadratarum ſeu duarum dimenſionum limitibus.

Prop. 1. $xx - lx + mm \propto 0$.

PEr tranſpoſitionem erit $mm \propto lx - xx$, & ſi prima pars fuerit realis, erit etiam altera pars realis, ideoque lx majus quam xx; & diviſo utroque termino per x, erit l major quàm x. Quin & per tranſpoſitionem propoſitæ æquationis habebitur $xx \propto lx - mm$: ideoque altera pars eſt realis, & lx majus quàm mm. Unde diviſo utroque termino per l, erit x major quàm $\frac{mm}{l}$. Quare æquationis propoſitæ utraque radix x major erit quàm $\frac{mm}{l}$, ſed minor quàm l.

Prop. 2. $xx - lx - mm \propto 0$.

Per tranſpoſitionem habebimus $xx \propto lx + mm$, ideoque xx majus erit quàm mm, & x major quàm m, ac proinde mx majus quàm mm. Unde xx minus erit quàm $lx + mx$, adeoque ſi utraque pars dividatur per x, erit x minor quàm $l + m$. Rurſus, quoniam xx æquatur $lx + mm$, erit xx majus quàm lx; ac proinde ſi uterque terminus dividatur per x, erit x major quàm l, & lx majus quàm ll. Hinc cum xx æquetur $lx + mm$, erit xx majus quàm $ll + mm$, hoc eſt, x major quàm $\sqrt{ll + mm}$. Poſtremò, quandoquidem x major eſt quam m, erit lx majus quàm lm, & xx majus quàm $lm + mm$, hoc eſt, x major quàm $\sqrt{lm + mm}$. Unde radix æquationis propoſitæ erit major quàm maxima harum duarum $\sqrt{ll + mm}$ & $\sqrt{lm + mm}$, ſed minor quàm $l + m$.

Q Prop.

Fig. 15. Two pages from Descartes, De Limitibus AEquationum (1661).

It is true that there was a great variety in the types of equation incorporated into the different works. Some of the authors placed pure quadratics as well as simple linear equations into one class and the affected quadratic equations, together with those of the cubic and biquadratic equations, into the compound composite class. All of these classifications, no matter how they were subdivided, can be reduced to the simple linear, pure quadratic and affected quadratic equations.

2. Symbolism and Terminology

a. SYMBOLISM.

The Rule of Algebra indicated, in many cases, the various methods of writing equations by considering the relationships between quantities and the methods of designating them. With some of these methods we are already familiar. After we briefly review the symbolism employed, we shall be better able to appreciate Glaisher's statement that

> No one could have imagined that such "trumpery tricks of abbreviation"-------could have led to the creation of a language so powerful that it has actually become an instrument of research.

The abbreviations used for plus and minus (piu and meno) were many and varied. The plus $\bar{p}$ and the minus $\bar{m}$ of Pisa, Paciolo, Cardan and Tartaglia were used by Chuquet and consequently by de la Roche, Feliciano, Gemma Frisius, Nuñez, Visconti and Cataldi; p. and m. by Peletier, Bombelli, Lauremberg and Unicorno; P. and M. by Ghaligai, Buteo and Gosselin; $\dot{+}$ and $\dot{-}$ by Graffenried, Follinus, Faulhaber, Ceulen and by Sesen. The latter author is not mentioned by Cajori or by Sister M. Leontius.[95] Faulhaber also employed $\div$ for minus and ✱ for plus. His use of these symbols was also omitted by Cajori in his Notations.

Peter Roth used $\dot{+}$ and $\div$. Brasser used the modern notation for plus, but his negative quantities were represented by a line with a short dash above and below it instead of the dots mentioned above, thus $\dot{-}$. Ghaligai used p or e for plus and me. or m^{o} for minus. Antich Rocha abbreviated plus into mas., meaning "many" and minus in me. from menos, meaning "less in a lower degree." The earlier writers retained the words piu and meno; as late as 1666 Bassi used them exclusively. Authors not mentioned here used the modern notation entirely.

The unknown quantity had a variety of titles, as GLORIOSI (1639) so aptly puts it,[96]

> Qui de Algebra scripserunt, terminos artis proprios, numeros scilicet illos Geometrice proportionales varijs nominibus appellarunt, generaliter a' plerisque dicuntur quantitates, dignitates, species, ab alijs vero numeri cossici, denominati, figurati, Franciscu Vieta nobilis Algebrista cum Diophanto Potestas vocati: sed non erit inutile horum terminorum secundum viam communem, & charactores adnotare, ut illico appareat quomodo inter se conueniant.

The symbols for the unknown were equally as varied. Ortego used the word cosa unabbreviated. Juan Diez and Bassi used cos, Ghaligai, c^{o}; co. was the common property of Paciolo, Cardan, Feliciano, Rocha, Moya, Visconti, Cataldi, Vav-Lezard and Unicorno. Leonardo of Pisa designated the unknown by the word radix. Scheubel and Graffenried abbreviated it to ra; Lauremberg, Renaldini and Klebit to R. and Tartaglia, Peletier, Henrion, De Billy, Chuquet, de la Roche, Faulhaber and Brucaeus to Ry. Grammateus and Van der Hoecke employed pri. for primus while Schagen (1628) used the initial letter only. The remaining symbols, together with the authors who used them are tabulated here.

$\mathcal{Z}_{\mathcal{L}}$ Recorde, Tap
ρ Buteo
$\mathcal{R}$ Riese, Rudolff, Stifel, Aurel, Petri,

[95] Sister M. Leontius Schulte, Additions in Arithmetic, Edwards Brothers, Ann Arbor, Michigan, 1935.
[96] Exercitationum Mathematicarum Decas Tertia, p. 44-----Gloriosi, Exercitationum.

Clavius, Ceulen, Sesen, Follinus, Brasser, Peucer, Wallis, W. P(hillip), Menher, Roth.
Digges
Stevin, Stampioen, van der Huips
Girard
Bombelli
Chuquet, de la Roche, Visconti
Salignacus, Oughtred
Cataldi
L. Gosselin
A Oughtred, Moore, Vasset
a Harriot, Herigon, Gibson, Kinckhuysen
i Hume
N Grisio, Xylander, Gloriosi, Vieta
x Des Cartes, van der Huips, Brasser, van Schooten, Kinckhuysen and Wallis

The representation of the second and third unknowns, if any were used at all, was rather more general. Besides the use of the word quantity or quan. for the second unknown, as employed by Nuñez and some of the earlier writers, the symbols were as follows:

2^a Paciolo
q Paciolo, Aurel, Rocha, and Rami
Rx Faulhaber
a Moya
A,B Peletier, Recorde, Buteo, Bombelli, Petri, Clavius, Ceulen, Henrion, Cataldi, Menher, Follinus, Hume, De Billy, Stampioen, van der Huips, Brasser, Renaldini
x,y Descartes, van Schooten, Kinckhuysen

TROPFKE says the following concerning the introduction of x, y, z:[97]

In explaining his algebraic calculations, an anonymous author of a Calculus (1638), a contemporary of Descartes, uses x in equations for two unknowns, x and y; only in one instance, where Descartes' own given equation employs a z, does z appear. In a collection of mathematical fragments written by Descartes, dating from the years 1629-1640, z occurs as the unknown only once; otherwise x and y always occur. There is still extant a fragment of the so-called Cartesian ovals, dated before 1629, in which x alone is introduced as an unknown, and y as a parameter, as we now say. This is, up to this time, the earliest instance in which one of the last letters of the alphabet appears in Descartes as the symbol for the unknown. It is characteristic that this is not a z. The practice of keeping the last letters of the alphabet especially for the unknown quantities was therefore at this time not yet conceived, but perhaps, as the accompanying y proves, was already germinating. A little later Descartes still used all three letters, x, y, and z, freely as known quantities. That x was preferred in the earliest instance can be explained, if one does not in fact wish to forego the pleasure of an explanation, only from its similarity to the cossic symbol 2ϱ.

The second power of the unknown was spoken of as censo or census by Ortego and Diez and as zens by Graffenried. Bassi (1660) used quadrati. It was otherwise represented by the following symbols which did not always correspond to those used for the first power.

ce. Paciolo, Cardan, Tartaglia, Feliciano, Rocha, Moya, Visconti, Cataldi, and Unicorno.
pri. Scheubel.
se. Grammateus and van der Hoecke.
Ghaligai, Recorde, Petri, Clavius, Ceulen, Sesen, Follinus, Brasser, Peucer, Coutereels, Wallis, Masterson, W. P(hillips), Tap, Menher, Brucaeus, Faulhaber, Roth
Peletier
Cataldi
Digges
bb Foster
(2) Girard
Bombelli
Stevin, Stampioen, van der Huips
q Salignacus, Henrion
Q Vav-Lezard, Oughtred, De Billy, Lauremberg, Grisio, Renaldini, Gosselin, Leybourn, Vasset, Xylander, Vieta
a2 Herigon
Aq Oughtred, Moore, Vasset
AA Rahn, Wallis, Faulhaber
Hume
R^2 Chuquet, de la Roche
1^2 Chuquet

Pictorial representation of the square as ▭ and ◇, of the cube ▭▭ and ▭, was given by Ghaligai and Buteo, respectively.

MISS VERA SANFORD in her Short

[97] Geschichte der Elementar-Mathematik, 1921, II, pp. 45-6. Author's translation.

History of Mathematics[98] mentions that the symbols employed by Bombelli and Stevin were "written above the coefficient." In the editions that we examined we found that they were written at the side of the coefficient.

The present sign of equality, the invention of Robert Recorde (1557), came into general use much later, through Harriot, Oughtred, Vav-Lezard, van der Huips, Brasser, Gibson, Foster, Rahn, Moore, Leybourn, Vasset, Wallis, though as late as 1678 Gottigneis used _ ∧ _ as the sign of equality. Cardan left a space between the terms or members equated. Most of the earlier authors simply wrote the word in full, or abbreviated it into ae, y, etc. Often the word and or the symbol & was used. Another symbol,

[was used by Paciolo and Buteo and Lauremberg. Cajori does not mention Lauremberg in his Notations
⌘ Digges
2|2 Herigon
=== Brasser. (This is similar to Ghaligai's)
· Follinus, (used only once)
} Descartes, Kinckhuysen
‖ Vav-Lezard

b. TERMINOLOGY:

DIGGES, on page 51 of Stratioticos interprets the word, Amasias,

Stifelius for aide of memory in this one word Amasias, representeth the rules of these 3 AEquations, by A or Ha, we may remember the first halfe or Moytie, the foundation of all operations, & therefore may be termed the Moytie Fundamentall, M signifieth Multiplication of that Moytie in itself, which I name squaring. A & S Addition and Subtraction, viz. in the two former AEquations, to adde the Square of the Moytie to the abstract number, in the last to Substract the one from the other, I, signifieth the Invention of the Square Root of the Product or Remaine, then A & S again admonisheth Addition or Subtraction of this root to be made to or from the moity Fundamentall, in forme as the Signe of that Moytie declareth.

STEVIN after stating problem LXVIII gives a "Nota."[99]

Le binomine du second terme donne de ce probleme se peut recontrer en trois differences, à scavoir:

① +⓪ Desquelles les autres an donnent
- ① +⓪ trois diverses operations, ausquels
① -⓪ Michel Stiffle a commode ce mot Amasias.

Cardan in Book X briefly expressed the above thus:

Querna, dabis.Nuquer, admi.Requan, Minue dami.

Antithesis, which has escaped the attention of the authors of the New English Dictionary, hereafter referred to as N.E.D., is explained by the Englishman Harriot (1631):[100]

Aequationum irregularium ad formam legitimam reductiones exemplificatae:
Per antithesim sive particularium transpositionem, quae sit per communem additionem.
Sit——aa - dc = gg——aequatio reducenda addatur utrinque—— + dc
Hinc sit——aa = gg + dc——reducta.

Harriot's idea of transposition, it will be observed, is not the modern notation of carrying a term from one member of the equation to the other and changing signs.

HERIGONE defines L'antithese as[101]

la transposition des quantitez de l'equation de l'une des parties à l'autre, leur donnant le signe d'affection contraire, en obseruant seulement, qu' aux quantitez données, les affirmées excedent celles qui sont niées. 3a2 bd + 2|2 2d2 ∼ fa,
ergo 3a2 + fa 2|2 2d2 ∼ bd.

RENALDINI (1644) calls Antithesis, the[102]

Reductione, vel Transpositione Aequationis.

Antithesis est numeri, seu quantitatis ex una aequationis parte, in alteram partem, sub contraria nota effectionis transpositio. It aut si quantitas aliqua afficiatur hoc signo ✱ , per Antithesin euadat affecta hoc nota -, & e contra.

[98] Vera Sanford, Short History of Mathematics, 1930, p. 156.
[99] OEuvres, 1634, p. 66.
[100] Artis Analyticae Praxis, p. 11--Harriot, Praxis.
[101] Supplementum Cursus Mathematici, 1642, p. 82--Herigon, Cursus.
[102] Opus Algebraicum, 1644, p. 133.

VIETA (Vav-Lezard's edition, 1646) says that[103]

L'antithese est quand les grandeurs affectantes ou affectees, passent d'une partie de lequation à l'autre sous contraires nottes d'affection.

GRISIO (1644), having shed much light on the history of mathematics by including in his text a list of authors who proved to be very influential, again summarizes the authors who have used some of the terms under discussion. He says,[104]

Nulla pur dico della regola dell' Algebra, e sue parti, nè tampoco dell' Antitesi, Parabolismo, Hypobibasmo, Isomeria, &c, si perche della prima cosa hanno trattato altri diffusamente, come lo Stifelio car. 227B; Lantz car. 88; Peletario car. 14. cap. 17. Clauio car. 36; cap. 8. Frà Luca car. 148A; & il Tartaglia car. 11; come perche il Sig. Mag. veramente non se l'attribuisce à sua inuenzione nè la pone per cosa nuoua; E delle seconde voci, e regole, parendo più tosto materia accattata dal Vieta, ...etc.

This leads to the discussion of transposition, to which antithesis is directly related. The first author of this period to use it is PELETIER (1554). He entitles his discussion "De la Transposicion des Sines Plus e Moins qui auient an l'Equacion."[105]

E sine de Plus, transpose: se conuertit an Moins: Le sine de Moins, transpose: se conuertit an Plus. Comme an notre Example, 4 ℔ p. 8, sont egales a 96: je transpose p. 8, qui deuiendra an m. 8, de l'autre part: ce seront, 4 ℔ egales a 96 m. 8: c'est a dire a 88. etc.

In the Whetstone of Witte (1557) by Recorde the Master expounds:

And evermore when occasion serveth, to translate nombers compounde, - on the one side is equal to + on the other side.
Scholar. Then it will be thus

$$84.\mathcal{Q}. + 38.\mathcal{Z}. = .10.\mathcal{Z}$$

Master. It were better thus

$$10.\mathcal{Z}. = .38.\mathcal{Z}. + .84.\mathcal{Q}.$$

And in smaller terms

$$5.\mathcal{Z}. = .19.\mathcal{Z}. + .42.\mathcal{Q}.$$

PEREZ DE MOYA uses the same idea without giving it a name, when he continues the example on p. 551.[106]

3 numeros. p. $\frac{1}{4}$ co. yg. a 6 numeros.

and then goes on to explain that by subtracting "3 nu.", which is in one part, from the 6 which is in the other, the result is "3 numeros" equal to "$\frac{1}{4}$ cosa."

DIGGES (1594) employs the words transport and transfer as well as transpose.[107]

Of transporting of numbers in AEquations.

This is a Rule generall, everie number transposed changeth his signe, as if I say 10 + 4 are equal to 3$\mathcal{Z}$ I may transfer 4 to 3$\mathcal{Z}$, I say 10 + are equal to 3$\mathcal{Z}$ - 4.——

This kind of Reduction by transportation, must be so ordered that you single by himself the greatest Character, so as the same may stand solitarie on the one side, and the lesser Characters from the contrary part of ȳ AEquation.

A similar explanation accompanies such terms as "transposer" used by Henrion, "transferier" by Graffenried, "translatione" and "transpositio" by Peucer, and "transportatur" by Lauremberg.

SESEN (1617) presents a clear conception of transposition,[108] as is demonstrated in the following literal translation of some passages from his work.

Therefore, in order to make the reduction of the equation more clearly understood, it should be started with the "restauration of the diminution," with the understanding that if the figure is accompanied by the sign ÷ it should be put on both sides of the equation, and, if the figure has the sign + and is carried over from one part (of the equation) to the other, it will have to be removed. The "restauration" or addition and transposition,

[103] Introduction en L'Art Analytic, ou Nouvelle Algebre de Francois Vieta, 1630, p. 54——Vav-Lezard, Introduction.
[104] Antanalisi, 1644, p. 107.
[105] L'Algebre, 1554, p. 26.
[106] Trattado, 1573.
[107] Stratioticos, p. 45.
[108] Flores Arithmetici, p. 746.

is carried out in the best manner by the rule of adding and subtracting. The entire art of "reduction" is contained in these two praeceptis:

The first Praeceptum. Everything that is transposed and carried over changes its sign. That is, if one part (of the equation) is accompanied by the sign ÷ and is carried over to the other part, the ÷ sign changes into + (and vice versa, + changes into ÷).

This is applicable in all previous "reductionibus." If first one part is carried over which would have the same denomination as the other part...and is going to be transposed, the following praecepti should be used.

The other Praeceptum. One sign for subtraction (reduction) and one sign for addition. That is, the part of the equation which is accompanied by the signs + ÷ would have on the other side a larger figure of the same kind and same sign. The figure of this part may be deducted from the other figure and the same sign remains as that which belonged to the figure from which it was deducted. Therefore, in carrying over to the other side, (the reduction) is started with the smaller figure. If the part which is going to be carried over to the other side (of the equation) has the same denomination with a contrasting sign, the figure of this part must be added....and one obtains the sign which belongs to the figure to which it is added.

In order to make it still clearer we show in the following (2) equations the reduction 6 ℓ ÷10 equal 10 ℓ ÷34. While the figures 10 and 34 have the same sign ÷, the smallest may be carried over; therefore 10 is deducted from 34 and the following equation results: 6 ℓ and 10 ℓ ÷24.

Furthermore, since the figures 10 ℓ and 6 ℓ have the same sign +, the smallest, 6 ℓ, must be deducted from the larger, 10 ℓ, in order to obtain the equation under 0 ℓ and 4 ℓ ÷24.

Finally, if one carries over ÷24 the equation is 24 ℓ and 4 ℓ.

Another example: 54 + 4 ℓ equal ʒ ÷6 ℓ +30 as the +4 ℓ and ÷6 ℓ have different signs, 4 ℓ and 6 ℓ are added together, resulting in ÷10 ℓ. The equation is then 54 and ℔ ÷10 ℓ +30. Since 54 and +30 have the same sign +, 30 is subtracted from 54 to obtain the equation ℔ and 10 ℓ +24.

Vieta's method is given by VAV-LE-ZARD in his edition of VIETA's work,[109]

Comme Ac - Bs estant esgal a DqZ, si Bs est transporte en l'autre part de l'equation; scauoir ----en + ac sera faict egal a DqZ + Bs par antithese.

MOORE is very explicit and gives a definition that is almost identical with that of Oughtred,[110]

That the magnitude or magnitudes sought may make up the one side of the AEquation by Transposition, you must convert the given magnitudes, and they sought from one side to the other, changing signs.

$$ac + bc = d.$$
$$ac = d - bc.$$

Although TWYSDEN does not define the term, he nevertheless uses the idea of transposition. After giving Z = A + E, he writes it again as Z - E = A, adding the phrase "by transposition."

JEAKE is credited with the word "transposition" in N.E.D. It was, however, used much earlier than 1664 by the following Englishmen: Recorde (1552), Digges (1579), Harriot (1631), Oughtred (1647), Twysden (1659) and Moore (1660).

The definition of the word appliation has also been omitted in N.E.D. We found it in only one work, The Key of the Mathematics, 1647, by OUGHTRED. As applied to algebra, it is there instanced thus:[111]

If the highest species of the Magnitudes sought, be drawn into some Magnitude given; let there be made unto it, a common Appliation of all the Magnitudes of the AEquation. As

$$BAq + BqA = Zc \qquad BA^2 + B^2A = C$$

It shall be,

$$Aq + BA = \frac{Zc}{B} \qquad A^2 + BA = \frac{C}{B}$$

Parabolism is the term applied to this same method of reduction by Vieta, Harriot, Herigon, Renaldini. Harriot's definition will serve to illustrate those given by the others.[112]

Per communem diuisionem, quam potestas a componente subgraduali liberature, quae est Vietae Parabolismus.

Sit——baa + dca = gcd——reducenda

Ergo—— $\frac{baa}{b} + \frac{dca}{b} = \frac{gcd}{b}$

Ergo——aa + $\frac{dca}{b} = \frac{gcd}{b}$ ——reducta.

[109] Introduction, 1630, p. 54. [110] Arithmetick, 1660, p. 58. [111] Key, 1647, p. 54. [112] Praxis, 1651, p. 11.

Fig. 16. Title page of Vieta, L'Algebre Nouvelle translated into French by A. Vasset (Paris, 1630).

ALGEBRE
DE VIETE,
D'VNE METHODE
NOVVELLE, CLAIRE,
ET FACILE.

Par laquelle toute l'obſcurité de l'Inuenteur eſt oſtee, & ſes termes pour la pluſpart inutiles, changez és termes ordinaires des Artiſtes.

DEDIE'
A Monſeigneur Claude Bouthillier, Cheualier, Conſeiller du Roy en ſes Conſeils, Commandeur, & Grand Treſorier des Ordres de ſa Maieſté, & Sur-Intendant des Finances de France.

A PARIS,
Chez Louys Boviengier, rue Sainct Iacques, à l'image Sainct Louys.

M. DC. XXXVI.
Avec Privilege du Roy.

Que l'egalité n'eſt point changée par le moyen de l'Hypobibaſme.

SECONDE PROPOSITION.

QVe A cube, plus B par A quarré, ſoit egal à Z plan par A. Ie dis que par l'Hypobibaſme A quarré, plus B par A, eſt egal à Z plan.

Car cela meſme eſt auoir diuiſé tous les ſolides par vn commun diuiſeur, par le moyen duquel l'egalité n'eſt point changee, comme il a eſté cy deuant determiné.

11. Et s'il arriue que le plus haut degré auquel la quantité incogneuë ſe trouue monter, ne ſubſiſte pas de ſoy meſme, mais eſtre multipliee par quelque grandeur, il ſe faudra ſeruir du paraboliſme.

Le paraboliſme eſt l'application commune de tous les Homogenes, dont l'æquation eſt compoſee, à vne grandeur donnee, par laquelle le plus haut degré de la quantité incogneuë, ſe trouue multiplice, affin que ce degré prenne la denomination de puiſſance, & que l'æquation ſubſiſte ſur icelluy; par le moyen dequoy l'egalité n'eſt pas changee. Ce qu'il faut en paſſant auſsi demonſtrer.

Fig. 17. A page from Vieta's L'Algebre Nouvelle showing the reduction of equations by hypobibasme.

Here we observe that this expression was defined as early as 1631, almost three quarters of a century earlier than its definition in the vernacular.

Another method of reduction is given by both Oughtred (1631) and Moore (1660), and is known as depression. OUGHTRED is the first one who used this term although the same idea was given earlier under the name of hypobibasmus. He says that[113]

-----if it happen that all the Magnitudes given be drawne into some degree of the Magnitudes sought; let there be made a Common Depression of them all, according to the order of the Table by adplying them to the least species. As Ac + XAq = Nc, divided by A, giveth

$$Aq + XA = \frac{Nc}{A} \qquad A^3 + XA^2 = N$$

But divided by Aq, it makes $A^2 + XA = \frac{N}{A}$

$$A + X = \frac{Nc}{Aq} \quad \text{or} \quad A + X = \frac{N}{A^2}$$

MOORE (1660) is somewhat clearer in his definition,[114]

If the Magnitude sought be found in all the magnitudes on either side of the AEquation; then you must by depression or dividing the magnitudes sought by some Parodical degree, clear some species of the AEquation, that the side may be known. Example: ac + baa = da, depressed becomes, aa + ba = d.

As indicated above this term has its equivalent in that of hypobibasme. VAV-LEZARD gives its definition[115]

Comme Eqqq + DEqc estant egal a DqEqq l'egale d'epresion affin qui demeure vn homogene donne, se feroit diuisant toutes les grandeurs par Eqq laquelle faicte restera Eq + DE‖ .Dq. Par l'hypobibasme l'egalité n'est changée.

HERIGON asserts that[116]

L'hypobibasme est un egal abbaissement de la puissance & de ses degrez paradiques, & se fait en sonstrayant le moindre degré parodique de la puissance & de tous ses degrez parodiques, qui se doiuent trouuer en toutes les parties de l'equation.

3a3 + abd 2|2 2ad2 ∼ fa2,
ergo 3a + bd 2|2 d2 ∼ fa.

Renaldini has the same definition. HARRIOT describes this method of reduction,[117]

Per communem diuisionem, qua homogeneum datum a componente graduali liberatur, quae est Vietae Hypobibasmus.

Sit-----aaa + baa = dca-----reducenda

Ergo----- $\frac{aaa}{a} + \frac{baa}{a} = \frac{dca}{a}$

Ergo----- aa + ba = dc-----reducta

DE BILLY (1637) asks the question "Qu' est-ce que l' hypobibasme, & comment se fait-il?" This he answers:[118]

Hypobibasme n'est rien autre chose qu'vn rabais on depression de caractere, & il se fait par la subtractio du moindre exposant: comme si j'avois trouue equatio 10 QC, & 90QQ, je regarderois l'exposant de QC, etc.

Girard and Herigon speak of simplifying an equation of fractional terms as Isomerie. Both definitions are essentially the same; namely, that it is the reduction of fractions of the same denomination by multiplying each numerator and also the whole numbers, if there are any, by the denominators of the other fractions, or by a number which is divisible by all the denominators "comme nous auons dit aux fractions de l'Arithmetique." HERIGON furnishes the example:[119]

$$a + \tfrac{7}{12} \quad 2|2 \quad 4 + \frac{5a}{8}$$

$$\text{ergo } 24a + 14 \quad 2|2 \quad 96 + 15a$$

HUME'S Traicte de l'Algebre introduces a new term, Logarymes, in chapter VII,

[113] Key, 1647, p. 55.
[114] Arithmetick, p. 59.
[115] Introduction, 1630, p. 56.
[116] Cursus, 1634, p. 82.
[117] Praxis, 1631, p. 11.
[118] Abrege Des Preceptes D'Algebre, p. 13.
[119] Cursus, 1634, p. 81.

entitled "De l'Equation, & le reduction d'Equation." He says,

La reduction se fait seulement quand le nombre qui a le plus grand logaryme ne se trouve pas seul; comme si il y avoit une equation entre 8ij + 3, & 16 + 5j. Item quand le plus grand logaryme se trouve en tous les deux termes de l'equation, comme en 5ij, & 16ij + 2j, cu le plus grand logaryme ij se trouve en tous les deux nombres.

FOLLINUS, in his Regula Algebrae, records the possible forms an equation may assume. The second form he calls an identity, giving an example which results in 100 being equal to 100. He explains further that[120]

Quare vbicunque aequatio inuenitur inter duos numeros Cossicos aequales, euisdemque denominationis, quaestio proposita per quemuis numerum soluitur, vnde & Nugatoria appellatur.

CLAVIUS speaks of "quaestio possibilis, sed nugatoria," giving examples to illustrate his explanation. One is as follows:[121]

Item numerus 10. diuidendus sit in tales duas partes, vt numerus, qui sit ex vna parte in totum numerum, aequalis sit quadrato eiusdem partis, vna cum numero, qui ex eadem parte in alteram ducta procreatur.

Ponator prima pars esse 1♀. ideoque altera 10 - 1♀. Et quia ex priori parte, id est, ex 1♀. in totum numerum 10. fiunt 10♀: Ex eadem autem parte in se fit 1♂. & ex 1♀ in 10 - 1♀ fiunt 10♀ - 1♂. quae addita ad 1♂. fiunt 10♀. quia vt 1♂ addatur ad - 1♂. debet fieri subtractio, vt in additione dictum est. Igitur inuenta est aequatio inter 10♀. & 10♀. quae similiter nugatoria est: fierique potest, quod proponitur, in quolibet numero, quomodocunque in duas partes distributo, vt demonstratum est ab Eucl. lib. 2. propos. 3. Quod si altera pars 10 - 1♀ ducatur in totum numerum 10♀. fiunt 100 - 10♀. Si vero eadem pars in se ducatur, fiunt 100 - 20♀ + 1♂. & duae partes 1♀ - & 10 - 1♀ inter se multiplicatae faciunt 10♀. & 10 - 1♀ inter se multiplicatae faciant 10♀ - 1♂ quae cum 100 - 20♀ + 1♂. faciunt 100 - 10♀. quae aequalia esse debent 100 - 10♀. producto scilicet secundae partis in 10. quae aequatio nugatoria quoque est.

RENALDINI (1655), as well as FOLLINUS (1622), indicates a problem as "vana et nugatoria."

Fourteen years earlier than Jeake, who is cited in N.E.D., MOORE (1660) says concisely that "if the AEquation fall out contrary to the definition then it is either Nugatory or impossible":[122]

1) As if there be an Identical AEquation (in the original, Aquation) thus, 3Aq = 3Aq. Then may it be resolved by any number whatsoever, and therefore it is a vain question:

2) If the AEquation fall out to be 5Aq = 4Aq. Then is the question propounded utterly impossible, and cannot be resolved: Likewise if the question be Aq = 12A - 40 because 40 cannot be taken from 36 the square of 6; $\frac{12}{12}$: there likewise is this question impossible to be answered: and thus much for the work accompanying the AEquation.

It is worthy of attention here that a negative root was considered as an impossible answer. Each one of the authors who used the term Nugatory also used the term impossible for negative roots.

RENALDINI used terms which we have found nowhere else.[123]

Potestas autem hoc modo se habens, Affecta dicitur. Haec verò affectio, vel est per affirmationem, vel per negationem. Affectio per affirmationem sit per signum ✱ diciturque; Cataphatica affectio: per negationem sit per signum ——, & Apophatica nuncupatur; ac verò Amphibola, seu ambiqua, cum dignitati maiori, siue Potestati praefigitur signum ——: quoniam haec radix duplicem habet radicem, ideo Ambigua dicitur.

In NICHOMACHUS OF GERASA we find a number of terms for expressing certain conditions in problems which were similarly defined by Gemma Frisius, Scheubel, Recorde, Clavius, Visconti, Wallis, and others. One is superparticular, which Nichomachus defines as[124]

the second species of the greater both naturally, and in order, is a number that contains within itself the whole of the number compared with, and some one factor of it besides.

If the factor is a half and the greater of the terms compared is called specifically sesquialter, and the smaller subsequialter, ...

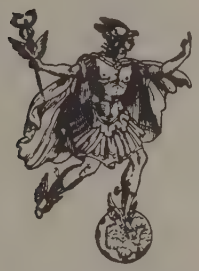
Superpartient is the relation when a number contains within itself the whole of the number

[120] Algebra, 1622, p. 181.

[121] De Algebrae bound with Georgii Henischii B. Medici et Mathematici Aug. Commentarius in Sphaeram Procli Diadochi, 1609, p. 71. [122] Aritnmetick, p. 61. [123] Opus Algebraicum, 1644, p. 123.

[124] D'Ooge, Robbins, Karpinski, Nichomachus of Gerasa, 1926, pp. 220-229.

L'ALGEBRA
OPERA
Di RAFAEL BOMBELLI da Bologna
Diuisa in tre Libri.
Con la quale ciascuno da se potrà venire in perfetta cognitione della tecnica dell'Arimetica.
Con vna Tauola copiosa delle materie, che in essa si contengono.
Posta hora in luce à beneficio delli studiosi di detta professione.



IN BOLOGNA,
Per Giouanni Rossi. MDLXXIX.
Con licenza de' Superiori.

Fig. 18. Title page of Bombelli, L'Algebra Opera (Bologna, 1579).

ALGEBRA
CHRISTOPHORI
CLAVII
BAMBERGENSIS
E SOCIETATE
IESV.

ROMAE,
Apud Bartholomæum Zannettum.
Anno M. DC. VIII.
SVPERIORVM PERMISSV.

Fig. 19. Title page of Clavius, Algebra (Rome, 1608).

4.p.R.q.L 24.m.20. J Eguale à 2.
R.q.L 24.m.20. J Eguale à 2.m.4.
24 m. 20. Eguale à 4.m.16.p.16.
24. p. 16. Eguale à 4.p.20.p.16.
24 Eguale à 4. p. 4. p.16.
8. Eguale à 4. p. 4.
2. Eguale à 1. p. 1.
2 ½ Eguale à 1.p.1.p.¼
1 ½ Eguale à 1. p. ¼
1. Eguale à 1.
R 2 Aggua-

Fig. 20. Part of a page from Bombelli, L'Algebra (1579).

esser in 8 — 8, sic vt productus 16 — 16. sic vero productio maior. Quare ex multiplicatione — 3, in 8. fiunt 24. notanda hoc signo — atque ita rursus ex — in + sit —. Sed quia non eorum hoc, quod prouenit ex — 3. in 8. sed quod producitur ex — 3. in 8. subtrahi debet prius 2. idcirco restituendum est id, quod iniuste ablatum est; atque ita postremo ex — 3. in — 2. fiunt 6. notanda signo hoc +, quod additionem Epsilicae, vt restituantur 8. quæ ablata fuerant. Talem ratio est in cæteris exemplis Multiplicationis.

IAM vero cum ex multiplicatione 8. in — 4. prodeat hic numerus — 24. vt dictum est, peccatic est, vt ex diuisione — 24. per — 4. producatur 6; & ex diuisione — 24. per 6. fiant — 4. Quare in diuisione quoque ex —, in —, sit +, & ex —, in +. vel contra + —, itaq. quemadmodum affirmatio affirmationis, & negatio negationis affirmat, ita siue Plus in Plus, siue Minus in Minus multiplicatum, aut diuisum producit Plus. Item sicut affirmatio negationis, siue negatio affirmationis negat, ita siue Plus in Minus, siue Minus in Plus multiplicatum, aut diuisum producit Minus.

DE NVMERIS FICTIS, SIVE MINORIBUS quam nihil. Cap. VI.

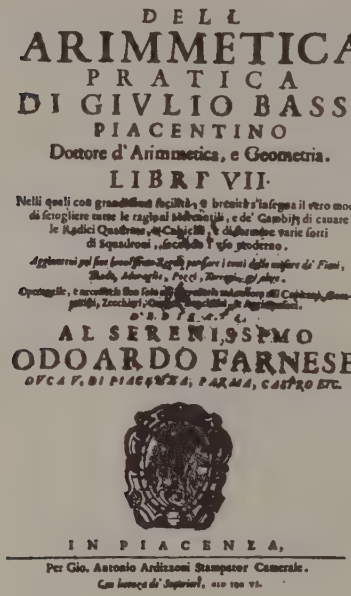
Numeri fictitij, per quos minores quam nihil.

QVEMADMODVM varia singuntur radices numerorum, qui eas non habent, qualis est radix quadrata, vel cubica, vel Zensicensica, vel Surdesolida huius numeri 2, siq. hæc fictis numeris vtiliter, & commode eorum, qui in rebus Mathematicis versantur, vt suo loco dicemus: Ita etiam a scriptoribus Algebræ finguntur non temere numeri quidam minores quam nihil, vt 0 — 4. hoc est, nihil detractis prius quatuor vnitatibus. Item 8 — 10. hoc est, ex vnitatibus minus 10. vnitatibus. Quod vnico exemplo hæc fictionis perspicuum fiet. Sint ex 8 — 2, deducenda 10 — 5. hoc est,

8 — 2
10 — 5
— 2 + 3

5, ex 6. Primo detrahi — 5 ex — 2, hoc est, quia superant numeri minore est, detrahi 2, ex 5, & reliqui numero 3, tribus constantium signum +, iuxta legem fubractionis supra traditam. Nam re vera hic numerus — 2. intelligitur superare hunc numerum — 5, tribus vnitatibus, cum — 2, cadat infra nihil duobus vnitatibus, at — 5. infra idem nihil descendat quinque vnitatibus, hoc est, tribus vnitatibus magis, quam — 2, recedat a nihilo. Quod etiam sine pereſt intelligi, quod huic numero — 2, desint tantum duæ vnitates, vt æquatis se nihilo, at huic — 5, desint quinque, vt nihilo æquiualeat. Deinde subtracto + 10, ex + 8, maiorem numerum ex minore, quod fieri non potest. Vade non inuenio numerum aliquem

Fig. 21. A page from Clavius, Algebra (1608).



DELL'
ARIMMETICA
PRATICA
DI GIVLIO BASSI
PIACENTINO
Dottore d'Arimmetica, e Geometria.
LIBRI VII.

Nelli quali con grandissima facilità, & brevità s'insegna il vero modo di sciogliere tutte le ragioni Mercantili, e de' Cambij di cauare le Radici Quadrate, & Cubiche, & diuersamente Parte sotti di Squadroni, secondo l'vso moderno.

Aggiuntoui poi sua breuissima Regola, per sapere i conti delle misure de' Fiumi, Boschi, Arborghi, Pozzi, Torrioni, & altre.

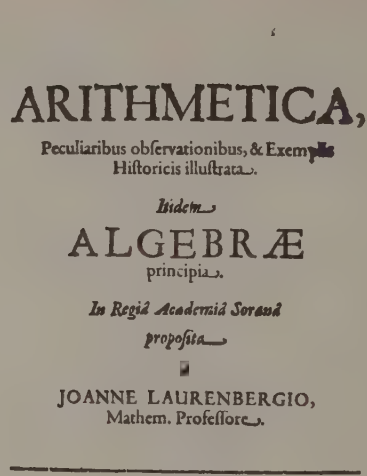
Opera utile, & necessaria a ogni sorte di persone studiose nella scientia del Contare, Compartire, Tenchieri, Cassieri, Scrittori, & Agrimensori.

AL SERENISSIMO
ODOARDO FARNESE
DVCA V. DI PIACENZA, PARMA, CASTRO ETC.

IN PIACENZA,

Per Gio. Antonio Ardizzoni Stampator Camerale.
Con licenza de' Superiori, & con gli Vti.

Fig. 22. Title page of Bassi, Dell' Arimmetica Practica (Piacenza, 1606).



ARITHMETICA,

Peculiaribus obſervationibus, & Exemplis
Hiſtoricis illuſtrata.

Iidem

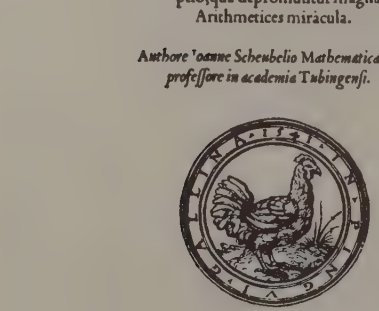
ALGEBRÆ
principia.

In Regiâ Academiâ Soranâ
propoſita

JOANNE LAURENBERGIO,
Mathem. Profeſſore.

SORÆ,
Excudit HENRICUS CRUSIUS,
Acad. Typogr.
ANNO ↄ Ↄ C XLIII.

Fig. 23. Title page of Lauremberg, Arithmetica (Sorae, 1645).

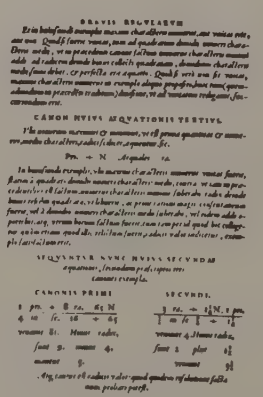


ALGEBRAE
COMPENDIOSA
FACILI'SQVE DESCRIptio, qua depromuntur magna Arithmetices miracula.

Authore Ioanne Scheubelio *Mathematicarum professore in academia Tubingensi.*

PARISIIS,
Apud Gulielmum Cauellat, in Pingui Gallina, ex aduerso Collegii Cameracensis.
1551.
CVM PRIVILEGIO.

Fig. 24. Title page of Scheubel, Algebrae Compendiosa (Paris, 1551).



BREVIS RESOLVTIO

CANON HVIVS AEQVATIONIS TERTIVS

Pro + N Aequales

SEQVITVR NVNC HVIVS SECVNDA

CANONIS PRIMI SECVNDI.

Fig. 25. A page from the same text dealing with quadratic equations.

compared and in addition more than one part of it.

Subsuperpartient immediately is produced, whenever a number is completely contained in the one compared with it. ...

If 9 is superpartient of 7, being $1\frac{2}{7}$ of it then 7 is contained in 9, $1\frac{2}{7}$ times and is a subsuperpartient of 9.

The words "continens" and "contentum" will be defined in the explanation of a problem by Buteo in the next section.

- - - - -

3. Reduction of Equations

a. SIMPLE LINEAR EQUATIONS:

SCHEUBEL (1551) illustrates the use of such ratios as were mentioned in the last section under the "Exemplum secundum,"[125]

Dividentur 40 in tres partes secundum rationem subsuperbipartientem tertias continuates.

Facit $7\frac{17}{49}$ $12\frac{12}{49}$ $20\frac{20}{49}$

Operatio.

Esto 1 ra. prima	x, the first part
quare $1\frac{2}{3}$ ra. secunda	$1\frac{2}{3}$ x = the second part
ac $2\frac{7}{9}$ ra. deinde, tertia pars erunt.	$2\frac{7}{9}$ x = the third part
Summa igitur $5\frac{4}{9}$ ra. aequales 40. N.	$5\frac{4}{9}$ x = 40

If we solve for "ra" we find that the first number will be $7\frac{17}{49}$ and multiplying that by $\frac{5}{3}$, one will obtain $12\frac{12}{49}$ for the second; and again multiplying the last answer by $\frac{5}{3}$, one will find that it checks with the $20\frac{20}{49}$ given above.

RECORDE (1557) has the Scholar solve a problem as follows:

Master: This is another question. There is a grounds inclosed with .4. walles, beyng like tombes and of one heighte. The longest .2. walls are in proportion to the shorteste, as .5. to .3. And unto the height these bee double Sesquialter. Now multipliyng the longest by the shorteste, and that totalle by the heighte, there will rise .39930. foote I demande then, what is the lengthe and the heighte of eache walle?

Scholar: The least quantitaie is the heighte, whiche I call .1.Z. and vnto it the longeste walle is double sesquialter: that is .$2\frac{1}{2}$Z. Now that same longeste is in proportion Superbipartiente quintas, to the shortest walle. So must the shorter wall be $1\frac{1}{2}$Z. Then must I multiplie all those .3. nombers together, that is 1.Z. by .$1\frac{1}{2}$Z. whereof doeth come .$\frac{3}{2}$ȝ. then shall I multiplie that totalle, by $\frac{5}{2}$Z, and it will be $\frac{15}{4}$ ȣ. or $3\frac{3}{4}$ ȣ which must be equalle, by the woordes of the question, to .39930.

Derivations of the expressions quoted above were used by these authors.

RECORDE in "The somme of the rule of equation" goes on to teach:[126]

When any question is propounded, appertaining to this rule, you shall imagine a name for the nomber, that is to be soughte, as you remember, that you learned in the rule of false position. And with that nomber you shall procede, accordyng to the question, vntil you finde a Cossike nomber, equalle to that nomber, that the question expresseth, whiche you shal reduce evermore to the leaste nombers. And then divide the nomber of the lesser denomination, and the quotient doeth answere to the question. Except the greater denomination doe beare the signe of some rooted nomber. For then must you extract the roote of that quotiente, accordyng to that signe of denomination.---------- Nowbeit, for easier alteration of equations. I will propound a few examples, bicause the extraction of their rootes, maie the more aptly bee wroughte. And to avoid the tedious repetition of these woordes: is equal to: I will sette as I doe often in woorke doe, a pair of paralleles, or Gemowe lines of an lengthe, thus: ═══ bicause noe .2. thynges can be moare equalle. And now marke these nombers.

1. 14.Z. —+— .15.ȣ ═══ 71.ȣ
2. 20.Z. —+— .18.ȣ ═══ 102.ȣ

In the firste there appeareth .2. nombers, that is 14.Z. —+— 15.ȣ. equalle to one nomber, which is 71.ȣ. But if you marke them well, you maie see

[125] Algebra, p. 141.
[126] Whetstone of Witte, 1557, p. Ge. iiij[r].

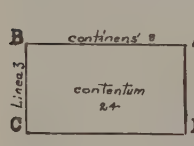
one denomination, on bothe sides of the equation, which neuer ought to stand. Wherefore abating the lesser, that is .15. ℘. out of bothe the nombers, there will remain, 14. 4 ═══ 56. ℘. that is, by reduction, 1 4 ═══ 4. ℘.

The symbol ℘ stands for the constant number. Farther on, after a short dialogue between the Master and Scholar on the axioms of equalities, he explains the function of the plus and minus signs.

....and if it be the signe of addition. —+—. then shall you abate that lesser nomber, from bothe the partes of the equation. As I did in this first example. But if the signe be of abatement ——, then shall you add the lesser nomber, to bothe partes. And so shall you doe, till there be noe one denomination on bothe sides, but diuerse and distincte.

This is the method generally employed in this period, although, of course, because of the late adoption of the sign of equality, most of the problems were not reduced to so concise a form. BUTEO directs us to operate with the second rule:[127]

Place the required number to be 1℘, "duc" 1℘ into 4, making 4℘. "Duc" 4℘ into 2, making 8℘. But here finally from the method above which is 8℘ I call "continens" which indicates "nota subsequens," as the principle of the rectangle, thus [, which will give in the end the proposed number, 24, which I called "contentum." And the same notation as before but concluding with the inverse rectangle in this fashion [24]. From this method it is understood to make a rectangle, ABCD, whose area we know from the above to be 24, containing the line AB which we know to be 8, multiplied in with the line BC. But the line BC which we have proposed from the beginning for the number sought, is 1℘. Such in truth is the number sought from the rule. Therefore divide the "contentum," 24, by the "continens" 8 which will result in 3, the number sought or the line BC.



He goes on to show that the result would be the same if 2℘ were taken for the unknown instead of 1℘.

It is to be observed here that he calls the sides of the equation continens and contentum, respectively, writing [between them if they are not reduced to simplest form. When the complete reduction has been made the contentum is inclosed in a complete rectangle, [ ]. A similar use of these expressions is made by Scheubel, Antich Rocha and Visconti, who also use the idea of "ductus."

SALIGNACUS simply says,[128]

Itaque aequatis 4ℓ & 88, concludemus	$4x = 88$
1ℓ & 22 aequari.	$x = 22$

CLAVIUS begins the study of equations by propounding,[129]

Find a number such that $\frac{1}{2}$ and $\frac{1}{3}$ of itself subtracted from it, leaves 7.

On pages 151-2 of Cajori's History of Mathematical Notations attention is called to a rare volume, KLEBIT's Insvlae Melitensis, qvam alias Maltam vocant, Historia, quaestionib. aliquot Mathematicis reddita incundior, Antwerp, 1565, containing exercises on the first degree equation in one unknown. On the page of this work which is here represented, one problem reads as follows:

Quot Triremes onustae equis ad hanc obsidionem vererint.

Si auferas ex munero nauium $\frac{1}{4}$ & $\frac{1}{2}$, restant 5. Quaestio est quot triremes fuerint?

	1 R. $-\frac{6}{8}$ aeq. 5
	$\frac{2}{8}$ R aeq. 5
Facit	1 R aeq. 20
Numerus quaesitus	

1 R. $-\frac{6}{8}$ means 1 R. $-\frac{6}{8}$ R., since 1 R. is equivalent to the x of modern notation.

FORCADEL, in the section on simple linear equations, fol. 69 of his Arithmetique (1556), includes a peculiar solution for the problem,

Find a number to which 7 is added and whose sum is multiplied by 6. 8 is then subtracted and after dividing the difference by 12, the result is $7\frac{1}{3}$.

[127] Logistica, 1559, p. 138.
[128] Arithmeticae, 1580, p. 134.
[129] Algebra, 1608, p. 162.

10 TRAICTE

De la reduction d'equation.

CHAP. VIII.

POVR l'intelligence de la reduction des equations, eſt à notter, que ſi on adiouſte ou ſouſtraict choſes égales de chaſque terme de l'equation, ou bien qu'on les multiplie ou diuiſe par vn meſme nombre, qu'il y aura pareillement equation entre les produits. Comme pour exemple, s'il y a equation entre 12 ℛ, & 72; oſtant 4 ℛ de chaſque terme, reſtera encore equation entre 8 ℛ, & 72—4ℛ: car puis que 12ℛ ſont egales à 72; 1ℛ ſera égale à 6, & partant 8ℛ ſeront égales à 48, & 4ℛ à 24, qui oſtez de 72, reſtera auſſi 48. Item ſi à chaſque terme de l'equation d'entre 8ℛ, & 72—4ℛ, on adiouſte 10, viendra equation entre 8ℛ+10, & 82—4ℛ. Car il eſt manifeſte que chaſque terme vaut 58. Item s'il y a equation entre 3ℛ+12, & 72—7ℛ; multipliant chaſque terme par 2, viendra auſſi equation entre 6ℛ+24, & 144—14ℛ: car l'vn & l'autre terme faict 60. Ainſi auſſi ſi nous diuiſons par 6 chaſque terme de ceſte derniere equation, ſera produit equation entre 1ℛ+4, & 24—2⅓ℛ: car l'vn & l'autre terme faict 10.

Maintenant quand en la ſolution de quelque queſtion on eſt paruenu à l'equation; ſi le plus grand caractere coſſique, par le nombre duquel (ſelon la regle d'Algebre) on doit diuiſer l'autre terme de l'equation n'eſt poſé ſeul en l'vn & en l'autre terme d'icelle equation, où qu'eſtant ſeul en l'vn, il ne le ſoit en l'autre, alors la diuiſion ne ſe peut faire. Comme pour exemple, ſi l'equation eſt trouuée entre 9ℛ+12, & 78—2ℛ; la diuiſion ne ſe peut faire, pource qu'en chaſque terme eſt trouué ce caractere ℛ, & ceſtuy-cy N: Semblablement ſi l'equation eſt trouuée entre 9ℛ—12, & 42, tu vois que la diuiſion ne ſe peut faire par 9 nombre du plus grand caractere coſſique ℛ, pource que 9ℛ ne ſont pas poſées ſeules, ains 9ℛ —12.

Ainſi auſſi ſi l'equation eſt trouuée entre 9ℛ, & 72—3ℛ: Item entre 1q—3ℛ, & 108: Item entre 1q—48, & 8ℛ: Item entre 108+8ℛ, & 2q—12ℛ +60, &c. en toutes leſquelles il eſt manifeſte que la diuiſion ne ſe peut faire, comme requiert la regle d'Algebre. Parquoy

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aduenant ſemblables equations, elles doiuent eſtre reduittes en autres, eſquelles le plus grand caractere coſſique ſoit ſeul en vn terme de l'equation, & ne ſoit repeté en l'autre, & eſquelles aucun caractere coſſique ne ſoit poſé deux fois. Or ceſte reduction ſe fera ainſi qu'il enſuit:

Si vne particule de l'equation a le ſigne —, il la faut tranſpoſer, c'eſt à dire adiouſter à l'autre terme: comme ſi vne equation eſt trouuée entre 9ℛ—12, & 42: nous adiouſterons 12 à 42, & nous aurons l'equation entre 9ℛ, & 54: Item l'equation eſtant trouuée entre 9ℛ, & 72—3ℛ, adiouſtant 3ℛ à chaſque terme, nous aurons l'equation entre 12ℛ, & 72: Item l'equation eſtant entre 2q—3ℛ, & 104, adiouſtant 3 ℛ à chaſque terme, l'equation ſera entre 2q, & 3ℛ+104: Item l'equation eſtant entre 5q—40, & 10ℛ; adiouſtant —40, nous aurons vne equation entre 5q, & 10ℛ+40. Mais quand vne particule a le ſigne +, il la faut ſouſtraire: comme ſi l'equation eſt trouuée entre 11ℛ+12, & 78, nous ſouſtrairons +12, & reſtera l'equation entre 11ℛ, & 66: Item l'equation eſtant trouuée entre 3ℛ+6, & 24; oſtant +6, reſtera l'equation entre 3ℛ, & 18: Item ſi vne equation eſt trouuée entre 5q+20, & 100, nous oſterõs +20, & reſtera l'equation entre 5q, & 80: Item vne equation eſtant entre 3q+2ℛ, & 56; oſtant +2ℛ, l'equation reſtera entre 3q, & 56—2ℛ. Que ſi l'vn & l'autre ſigne + & — ſont en vne equation, il faut adiouſter la particule du ſigne —, mais ſouſtraire celle du ſigne +. Comme ſi vne equation eſt trouuée entre 9ℛ+12, & 78—2ℛ: nous adiouſterons premierement —2ℛ, & l'equation ſera entre 11ℛ+12, & 78; puis ſouſtrayant d'icelle +12, reſtera l'equation entre 11ℛ, & 66: Item ſi vne equation eſt entre 5q—3ℛ, & 3q+20; nous adiouſterons 3ℛ, & l'equation ſera entre 5q, & 3q+3ℛ+20; & oſtant de ceſte-cy 3q, reſtera l'equation entre 2q, & 3ℛ+20: Item ſi vne equation eſt trouuée entre 108+8ℛ, & 2q—12ℛ+60: nous adiouſterons premierement 12ℛ, & l'equation viendra entre 108+20ℛ, & 2q+60; & d'icelle eſtans oſtez 60, reſtera l'equation entre 2q, & 20ℛ+48. Que s'il aduient quelque equation, comme entre 6ℛ—10, & 10ℛ—34, en laquelle les nombres 10, & 34 ont meſme ſigne —, il faut oſter le moindre nombre 10 de chaſque terme, & reſtera l'equation entre 6ℛ, & 10ℛ—24; & d'autant qu'en icelle les nombres 6ℛ & 10ℛ ont auſſi vn meſme +, il faut oſter le moindre 6ℛ du plus grand 10ℛ, &

Fig. 26. Method of the reduction of equations in Henrion's Traicte d'Algebra (1620).

After the usual preliminary explanation of the steps in the solution he gives us the following:

1 Ra.	egale	1 quan.	
1 Ra. + 7	egale	1 quan. + 7	
6		6	
6 Ra. + 42	egales	6 quan. + 42	12
8		8	$7\frac{1}{3}$
6 Ra. + 34	egales	6 quan. + 34	84
	88		4
6 Ra. + 34 egale à 88			88
+ 34		34	
6 Ra.	egales	54	9
1 Ra.	egale à	9	7
6 quan. + 34	egales à	88	16
34		34	6
6 quan.	egales à	54	96
2 quan.	egales à	18	8
1 quan.	egal à	9	88 par 12
			$7\frac{1}{3}$ par 1

One questions the purpose of all the details. Whether his pedagogical instinct led him to equate the unknown to some quantity, or whether his knowledge of algebra was insufficient, are questions whose answers have not been determined. The answer may be a combination of both reasons.

The first part of his solution tends to favor the first hypothesis. Since, however, he does not base the formulation of his equation directly on the conditions of the problem as they are stated, but arrives at the right-hand member of it by a marginal computation, one is inclined to accept the second hypothesis. This last conjecture is strengthened by the fact that the title page of Book II includes,

...premieres causes des egalissemens.

while that to Book III promises

...ensemble plusieurs questions, reigles, & demonstrations Mathematiques, auecques le propre subiect de l'Algebre. Le tout de l'inuention de l'Autheur.

In reality few pages are devoted to the subject of algebra.

MENHER, after an explanation of the very simplest of equations such as "3ʒʒ are equal to 27ʒ," gives a geometric demonstration of a simple linear equation,[130]

There is a quadrangular rectangle whose side A.B. is 8, and whose whole area is 32. How much is the side A.C.? Let it be 1ϱ multiplied with A.B. and it will give 8ϱ equal to 32. Dividing 32 by 8 will give 4, which is the value of an ϱ.

HENRION (1615) follows the style of Clavius. In a problem like

Trouuer vn nombre, duquel la tierce & quarte partie estans ostez, le nombre restant soit 10.

the explanation is as lengthy as that of Clavius, taking one page of fine print in order to arrive at the final solution, namely, that R, or x, is equal to 24. This is followed by a proof of his result:[131]

Parquoy la 4. partie de la regle d'Algebre n'a lieu an nostre exemple: mais quand, & quelle racine du quotient manifestera le nomber incogneu, il sera enseigné cy-apres. Maintenant si du nombre trouué 24 on prend $\frac{1}{3}$, scauoir est 8: puis $\frac{1}{4}$, scauoir est 6: icelles deux parties font ensemble 14, qui ostez des 24, reste le nombre 10, ainsi qu'il estoit requis en la proposition.

GRAFFENRIED borrows the first two examples from Rudolph's Coss: In the translation of the first one his symbolism is retained. (Arithmeticae Logisticae, 1619, 231)

Three boys have a sum of "gelts" to divide such that the first has $\frac{1}{3}$ the sum minus 2.gl. The other has a fourth part of the sum and 8.gl. The third took the rest and found that he had 1 gl. more than the first. The question is, how many "gelts" did they get? The entire sum is 1 Ra. The first has 1 Ra. - 2 (here - means minus). The other has $\frac{1}{4}$Ra. + 8. The third has the remainder. The first and second added together: $\frac{1}{3}$Ra: and $\frac{1}{4}$Ra: together make $\frac{7}{12}$ Ra: and 8. - 2. make 6. Thereupon the sum of the first and second makes $\frac{7}{12}$Ra. + 6 (the original text has 8 which must be a typographical error) and subtracting this from 1 Ra., the entire sum, leaves $\frac{5}{12}$ Ra. - 6 and this is the sum the third has. Now the third has 1.gl. more than the first. But the first has $\frac{1}{3}$Ra. - 2 which is $\frac{4}{12}$Ra. - 2. Adding 1.gl. to this gives $\frac{4}{12}$Ra. - 1. This is equal to what the third has. $\frac{4}{12}$Ra. - 6 equals $\frac{4}{12}$Ra. - 1. Reduce this equation into another so that it has no + or - signs. Take from each side - 1, leaving $\frac{4}{12}$Ra. equal to $\frac{5}{12}$Ra. - 5. Subtract from each side $\frac{4}{12}$Ra., leaving $\frac{1}{12}$Ra. - 5 on one side and nothing on the other. Then transpose 5, which is +5 since it was -5 on that side and the $\frac{1}{12}$Ra. equals 5. 12 times 5 gives 60. Then 1 Ra. equals 60. This is the sum of the "gelt."

The first has $\frac{1}{3}$Ra. - 2 which is 18, for $\frac{1}{3}$Ra. is 20 and taking 2 from that leaves 18.gl.

The other has $\frac{1}{4}$Ra. + 8 which is 23.gl. for $\frac{1}{4}$Ra. is 15 and 8 thereupon makes 23.gl.

The remaining third, $\frac{5}{12}$Ra. - 6 is 19.gl. Then $\frac{5}{12}$Ra. is 25. Taking 6 from it leaves 19. And thus is seen that the third has only 1.gl. more than the gl. of the first.

BOMBELLI in the chapter dealing with a "Tanti eguali a numero," or "a quantity or thing equal to a number," cites a variety of problems belonging to this type of equation. One problem reads,[132]

If one has 4 ⏝1 equal to 20, divide 20 by 4, giving 5 which is the value of the quantity. $4x = 20$ $x = 5$

Another interesting one is as follows:

4 ⏝1 p.2. is equal to R.q. ⌊8.p.R.q.2⌋. Taking 2 from each side leaves 4 ⏝1 equal to R.Q. ⌊8.p.R.q.2⌋ .m.2, $4x + 2 = \sqrt{8 + \sqrt{2}}$

[130] Practique ... avec la Regle de Ciffrer. & Geometrie, 1565, p. Siiij, - Menher, Practique. Author's translation.

[131] Les Quinze Livres des Elements D'Euclide, 1615, p. 309 - Henrion, Elements.

[132] Algebra, 1629, p. 241.

Fig. 27. Title page of Graffenried, Arithmeticae Logisticae (1619), and two consecutive pages of the third book.

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$\frac{144}{18} \div 84 \mid 7\ 15$
$\frac{288}{18} + 96 \mid 8\ \frac{16}{240}$ Theiler.

Procedier/vnd setz den Theiler Bruchsweiß vnder das Collect/ kommen $2\frac{1}{2}\frac{5}{100}$ heb auff in 3. vnd nachmahls in 5. kommen $2\frac{3}{16}\frac{5}{16}$ darauß Extrahier die Quadratwurtel/ kommen $\frac{1}{4}\frac{5}{4}$ oder $3\frac{1}{16}$ darvon numb $1\frac{1}{2}$ so du zuvor zu den Satzungen addiert hast/ Rest $2\frac{1}{4}$ so viel ist der kleiner Theil/vnd der gröſſer $7\frac{1}{4}$.

Die Sechste Regel Coß nach der Falsi.

ANämlich setze zwo Zahlen/ wie gewohnt/ vnnd Examinier sie nach der Frag/ was daran zu wenig oder zu viel kompt/ verzeichne.

Darnach mußt du bey dieser Regel insonderheit fleissig in acht nemmen/ die Cossischen vergleichungen/ als inn der fünfften Regel auch angezogen worden.

Wann drey quantiteten einandern vergleicht werden/ Also/ daß die gröſter vnnd kleiner gleich gesprochen werden der mittern/ Als Zensus + Dragma gleich Radices. Oder Cubus + Rad. gleich Zensus. Item Zensdezens + zens gleich Cubis, vnnd also forters/ so Subtrahier allemahl den halben Theil der mittern quantitet, vom jeder Satzung/ vnd multiplicier die Rest in sich selbs quadrate, was darvon kompt/ das setze an statt der ersten beyder Satzungen. Procedier alßdann ferrner nach der Falsi, vnd auß dem kommenden Quotienten/ Collectem/ ꝛc. Extrahier Radicem quadratam, solchen Radicem gib oder numb (nach gestalt der Fragen) den sibtrahierten halben Theil berührter quantitet, So zeyget dir das Collect oder Rest das recht vnd wahre Facit.

Auch merck/ daß durch die gantze Coß/ Dragma für die erste quantitet, Radix für die ander/ Zens für die dritt/ Cubus für die vierdt/ vnd Zensdezens für die fünffte/ Quantitet genennet wird/ vnd so forters ꝛc. Doch behalt hierinn gleichermeiß die hievor gegebne Anzeigung.

17. Item/

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17. Item/ Such ein Zahl/ wann ich erstlich 8. darvon Subtrahier/ darnach von der gefundener Zahl 6. Subtrahier. Die zwey Rest multiplicier mit einandern/ daß das Product vmb 4 mehr anzeyge/ dann die gefundene Zahl.

Sey/ die Zahl sey 10. darvon Subtrahier erstlich 8. darnach 6. Resten 2. vnd 4. die beyde Rest multiplicier mit einander/ kommen 8. die solten 10. + 4. Nemblich 14. seyn/ leugt — 6. Setz zum andern mahl/ die Zahl sey 12. Examiniers/ wie jen/ kommen 24. solten 16. seyn/ leugt + 8. Es gibt aber die Cossische solution bey diesem Exempel 1. Zens + 44. Dragma gleich 15. Radices. Darumb Subtrahier von jeder Satzung den halben Theil der mittern quantitet, als $7\frac{1}{2}$ Radices, Resten $2\frac{1}{2}$ vnd $4\frac{1}{2}$ die multipliciet quadratè, kommen $\frac{25}{4}$ vnnd $\frac{81}{4}$ Setz es ferner also zur Regel.

$\frac{25}{4} \div 6 \mid 2.\ 7.$
$\frac{81}{4} + 8 \mid 4.\ \frac{4.}{28.}$ Theiler.

Procedier nach allgemeiner Falsi, vnnd setz den Theiler/ vnd das Collect/ vnd erkleiner den Bruch in 7. kommen $\frac{1}{4}\frac{2}{7}$ darauß Quadrata Radix thut $\frac{1}{2}$ oder $3\frac{1}{2}$ darzu addier die $7\frac{1}{2}$ so du oben von den Satzungen gezogen hast/ kommen 11. die begehrte Zahl.

18. Item es kommen zwo Bäwrin vom Marckt/ haben zusammen auß Hünnern gelöst 64. Kreutzer/hat die ander 4. Hennen mehr gehabt dann die erst. Spricht die erst zur andern/ Auß deinen Hunern wolt ich 42. kr. gelöst haben. Antwortet die ander/ so hette ich auß deinen Hennen gelöst 24. kr. wie viel Hennen hat jede gehabt?

Sey/ die erste hab 20 vnd die ander 24. Hennen/ Rechne wie viel ein jede löset. Sprich auß 24. Hünern will die erste lösen 42. kr. Wie viel loset sie auß 20. Hünern? Facit $\frac{8}{1}\frac{4}{2}$ kr. Weiter

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which divided by 4, the coefficient of the quantity, gives

R.q. $\lfloor \frac{1}{2}$p.R.q. $\frac{1}{128} \rfloor$m. $\frac{1}{2}$

which is the value of the thing or quantity.

$4x = \sqrt{8 + \sqrt{2}} - 2$

$x = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{128}}} - \frac{1}{2}$

JOHN TAP, in The Pathway to Knowledge (1621), page 338, gives a Reduction of Equations similar to that in the text with the same title by W. P.(HILIPS) (1596). He says,

That you may the better understand the manner how to reduce Equations, it is no more then to know when 2 things are equall the one to the other, and if you adde to, or take from each part, like quantities, that still their proportions are equall, as for example. If 3 Pi. - 6 s. be equalle to 54 shillings, what is the value of one pound? For the resolution of this and such like questions, take this for a generall Rule, because the sign is -, adde to each part 6 s. that is to say, if you adde 6 s. to 3 Pi. - 6 s. there will be 3 Pi. and if you ad 6 s to 54 s. there will be also 60 s equal to 3 Pi. by which reason 1 Pi. is equall to 20 s.....

If 3 2ℓ + 5 bee equall to 14, subtract from each part 5 and there will rest 3 2ℓ equall to 9, or 1 2ℓ equall to 3.

In the Preface to this text, John Tap enlightens us as to the source of his work:

.....for the extraction of roots, I meane for finding the nearest root of a number not right Square or Cubicall, I have observed the method used in Gosselin upon Tartaglia, and for the Rules and questions in the Art of Cossicke Numbers, it is onely a literall translation collected out of Mr. Valentine Menher his Arithmetic.

The similarity between these two texts with the same title has led to the investigation of their sources. The result is cited in the section on Quadratic Equations.

STEVIN does not specialize in simple equations. In his preliminary material he simply states that if 1 ① (x) has the value of 4, 5 ① + 3 is 23, because 5 ① is 20, p 3 is 23.[133]

STAMPIOEN gives a very unusual solution for the problem[134] in which he wishes to divide 22 ① + 176, (22x + 176), into two parts in such a way that the first with $\frac{1}{5}$ of the other is equal to the other with $\frac{1}{3}$ of the first.

Let the first part be 1A

So that the second will be 22 ① + 176 - 1A

Then $\frac{1}{5}$ is $\frac{22 ① + 176 - 1A}{5}$

which added to one A becomes $\frac{22 ① + 176 + 4A}{5}$

The second is 22 ① + 176 - 1A

Increased by $\frac{1}{3}$ of first as by $\frac{1}{3}$A, this results in 22 ① + 176 - $\frac{2}{3}$A. This will be equal to the first expression.

Then 22 ① + 176 - $\frac{2}{3}$A equals $\frac{22①+176+4A}{5}$

Multiplying each by 5 it becomes

110 ① + 880 - $3\frac{1}{3}$A equals 22 ① + 176 + 4A.

Subtracting 22 ① + 176 from each leaves

88 ① + 704 - $3\frac{1}{3}$A equal to 4A.

Adding $3\frac{1}{3}$A to each side, the result is

88 ① + 704 equal to $7\frac{1}{3}$A.

By the rule of three, $7\frac{1}{3}$A is 88 ① + 704. Then 1A becomes 12 ① + 96 for the first and the second will be 10 ① + 80.

BASSI in the 1666 edition of his Arithmetica, e Geometria pratica, page 549, gives about fifty problems on the linear equation, particularly those that require the division of a given number into parts or the finding of a number under certain given conditions as illustrated below.

La prima spetie semplice è quando le cose sono eguali al numero; allora diuidesi il numero per le cose, e l'auuenimento sarà il valore della cosa; per esempio: Trouisi vn numero, che leuandone $\frac{1}{3}$, e quello moltiplicato per 18 dia di pradotto 42? Suppongasi, che il detto numero sia 1 cosa, pigliasene $\frac{1}{3}$, e sarà $\frac{1}{3}$ cosa, che

x = the number

$18(\frac{1}{3}x) = 42$

6x = 42

x = 7, or the number

[133] OEuvres, 1634, p. 65.

[134] Algebra, 1639, p. 35.

leuata dal 18, darà 6, e con questo diuidesi il 42, che ne verrà 7 per lo numero ricercato. Proussi, pigliando $\frac{1}{3}$ di 7, che sarà $2\frac{1}{3}$, il qual moltiplicato col 18, produrrà il 42.

Proof:
$\frac{1}{3}$ of $7 = 2\frac{1}{3}$
$18 \times 2\frac{1}{3} = 42.$

The Arithmetica of 1628 by SCHAGEN presents us with only linear equations in which he introduces "p" for the unknown. Neither Cajori[135] nor Sister M. Leontius[136] makes any note of Schagen or his symbols. His problems are of the type that if $3p = 12$ then $1p = 4$.[137]

The remaining authors, if they treat the simple equations at all, follow the general plan of the examples cited, with no outstanding peculiarity that would warrant further study of this type of equation.

b. SIMULTANEOUS EQUATIONS:

When more than one unknown quantity arises, as is shown in our summary of symbols, the first letters of the alphabet capitalized are generally used. Equations involving any number of unknowns are dealt with as simultaneous equations by Peletier, Buteo, Ceulen, Gosselin, Rahn, and Descartes, although such solutions do not bear the modern title. The first one in this period to give such a solution is PELETIER. The problem which he solved, and which belongs to the famous problem dealing with three men each having a sum of money, he obtained from Cardan.[138]

Troes hommes ont chacun vn nombre d' Ecuz: Le premier, avec la $\frac{1}{2}$ des deus autres, an à 32: le second, avec la $\frac{1}{3}$ partie des deus autres, an à 28: Le tiers, avec la $\frac{1}{4}$ partie des deus autres, an à 31: Combien an ontiz chacun?

Three men each have a number of coins; the first with one half of the other two has 32; the second with a $\frac{1}{3}$ part of the other two has 28; the third with the fourth part of the other two has 31. How many coins has each?

He assigns 1R to the first amount, 1A to the second, and the third becomes 31m. $\frac{1R\text{ p. }1A}{4}(31 - \frac{x + A}{4})$. He then continues the explanation of its solution for two pages, finally coming to the conclusion that the first man has 12; the second has 16 and the third, 24. But he goes on to explain that he has here employed Cardan's method step by step. Peletier follows this, however, by his own method, assigning 1R (x) to the first; 1A to the second, and 1B to the third; and because the first with $\frac{1}{2}$ of the other two was equal to 32, then by reduction and two transpositions 2R p. 1A p. 1B equals 64 which makes the first equation.

Secondly, because the second with $\frac{1}{3}$ of the other two equals 28, i.e., 1A p. $\frac{1R\text{ p. }1B}{3}$ equals $28(A + \frac{x + B}{3} = 28)$, then by reduction 1R p. 1B. p. 3A equals 84. $(x + B + 3A = 84)$. This is the second equation.

For the third (which with $\frac{1}{4}$ part of the other two makes 31) we have 1B p. $\frac{1R\text{ p. }1A}{4}$ equal to 31. By reduction 1R p. 1A p. 4B equals 124. This constitutes the third principal equation.

He then disposes of the three equations in this manner:

I.	2R p. 1A p. 1B equals 64	$2x + A + B = 64$
II.	1R p. 3A p. 1B equals 84	$x + 3A + B = 84$
III.	1R p. 1A p. 4B equals 124	$x + A + 4B = 124$

Joining the second and the third, we have for the fourth equation,

2R p. 4A p. 5B equal to 208 $\quad 2x + 4A + 5B = 208.$

Subtracting the first equation from the fourth, the difference gives a fifth equation,

V. 3A p. 4B equal to 144 $\quad 3A + 4B = 144.$

Adding the first to the second, we have for the sixth equation,

[135] F. Cajori, Notations, 1928.
[136] Sister M. Leontius Schulte, Additions in Arithmetic, Edwards Brothers, Ann Arbor, Michigan, 1935.
[137] Arithmetica, fol. 67.
[138] L'Algebre, 1554, p. 107.

VI. 3R p. 4A p. 2B
equals 148 $3x + 4A + 2B = 148.$

Adding the first to the third, the seventh equation becomes

VII. 3R p. 2A p. 5B
equals 188 $3x + 2A + 5B = 188.$

Adding these two last ones, the eighth equation results in

VIII. 6R p. 6A p. 7B
equal to 336 $6x + 6A + 7B = 336.$

Finally multiplying the third by 6 we have for the ninth equation,

IX. 6R p. 6A p. 24B
equal to 744. $6x + 6A + 24B = 744.$

And because the first two cossic numbers of these last two equations are alike, the difference of the numbers 336 and 744 (which is 408) is equal to the difference of 7B and 24B (which difference is 17B). Accordingly 17B equals 408 and by division 1B is equal to 24, the value of the third quantity. Because, in the fifth equation, 3A p. 4B is equal to 144, and 4 times 24 is 96, then 3A is equal to 48 and A is equal to 16, the value of the second quantity. Since $\frac{1}{2}$ the sum of the second and the third, which is 20, together with the first quantity gives 32, the value of the first is 12.

It is well to remember in interpreting the following problem, which BUTEO gives on page 190 of his Logistica qvae & Arithmetica vulgo dicitur (1559), that when he deals with positive terms the sign P for plus is omitted.

Find three numbers such that the first with a third of the remaining numbers makes 14. The second with a fourth of the others makes 8. The third with a fifth part of the first and the second makes 8. He goes on to say: "let the first be 1A, the second, 1B, and the third, 1C."

$1A, \frac{1}{3}B, \frac{1}{3}C$ [14	$1A + \frac{1}{3}B + \frac{1}{3}C = 14$
$1B, \frac{1}{4}A, \frac{1}{4}C$ [8	$1B + \frac{1}{4}A + \frac{1}{4}C = 8$
$1C, \frac{1}{5}A, \frac{1}{5}B$ [8	$1C + \frac{1}{5}A + \frac{1}{5}B = 8$

The author follows with these equations cleared of fractions.

3A . 1B . 1C [42	1^a	$3A + B + C = 42$, 1st
1A . 4B . 1C [32	2^a	$A + 4B + C = 32$, 2nd
1A . 1B . 5C [40	3^a	$A + B + 5C = 40$, 3rd

It will be observed that the comma and the period now do the work of the plus sign. He goes on to simplify by directing the multiplication of the second by 3, giving 3A, 12B, 3C [96 or $3A + 12B + 3C = 96$. Subtracting this from the first leaves 11B, 2C [54.

3A, 12B, 3C [96	$3A + 12B + 3C = 96$
3A, 1B, 1C [42	$3A + B + C = 42$
11B, 2C [54	$11B + 2C = 54$

Multiplying the third equation by 3 makes 3A, 3B, 15C [120 and subtracting from the first leaves 2B, 14C [78.

3A . 3B . 15C [120	$3A + 3B + 15C = 120$
3A . 1B . 1C [42	$3A + 1B + 1C = 42$
2B . 14C 78	$2B + 14C = 78$

Then multiplying this last result by 11 makes 22B, 154C [858, while multiplying 11B, 2C [54 by 2 will make 22B, 4C [108. Subtracting this equation from 22B, 154C [858, leaves 150C [750.

Dividing through by 150 gives 5, for the value of the third number, C.

Upon substitution of these values of C in the equation 2B, 14C, 78, 2B will be 8. Therefore 4 is the second number, B. Similar substitutions in the third equation result in 11 as the first number. Therefore the three numbers are 11, 4, 5.

GOSSELIN follows Buteo's method in the solution of simultaneous equations by omitting signs of any kind whatsoever for positive terms.

H. BOSMANS in Le "De Arte magna" de Guillaume Gosselin furnishes us with an example of simultaneous equations. The problem[139] reads,

> Let us find 4 numbers whose first with a half of the remaining ones gives 17; the second with a third of the others is 12; the third with a fourth of the others is 13 and likewise the fourth with a sixth of the others is 13.

Gosselin assigns A, B, C, D to the four unknowns, formulating the equations as

[139] Enestrom, Bibliotheca Mathematica, Vol. VII, 1906-7, p. 56.

follows:

1A $\frac{1}{2}$B $\frac{1}{2}$C $\frac{1}{2}$D aequalia 17 $A+\frac{1}{2}B+\frac{1}{2}C+\frac{1}{2}D=17$

1B $\frac{1}{3}$A $\frac{1}{3}$C $\frac{1}{3}$D aequalia 12 $B+\frac{1}{3}A+\frac{1}{3}C+\frac{1}{3}D=12$

1C $\frac{1}{4}$A $\frac{1}{4}$B $\frac{1}{4}$D aequalia 13 $C+\frac{1}{4}A+\frac{1}{4}B+\frac{1}{4}D=13$

1D $\frac{1}{6}$A $\frac{1}{6}$B $\frac{1}{6}$C aequalia 13 $D+\frac{1}{6}A+\frac{1}{6}B+\frac{1}{6}C=13$

Clearing of fractions he obtains,

2A 1B 1C 1D aequalia 34 $2A+B+C+D=34$
1A 3B 1C 1D aequalia 36 $A+3B+C+D=36$
1A 1B 4C 1D aequalia 52 $A+B+4C+D=52$
1A 1B 1C 6D aequalia 78 $A+B+C+6D=78$

Adding the last two equations he obtains,

2A 2B 5C 7D aequalia 130. $2A+2B+5C+7D=130$.

And subtracting this result from the first equation, the remainder is

1B 4C 6D aequalia 96. $B+4C-6D=96$.

Now the sum of the fourth and second equations becomes

2A 4B 2C 7D aequalia 114. $2A+4B+2C+7D=114$.

This subtracted from the first equation leaves

3B 1C 6D aequalia 80. $3B+C+6D=80$.

The result of adding the first and third equation is

2A 4B 5C 2D aequalia 88. $2A+4B+5C+2D=88$.

This subtracted from the first gives

3B 4C 1D aequalia 54. $3B+4C+D=54$.

Now multiplying 1B 4C 6D, which is equal to 96, by 3 results in

3B 12C 18D aequalia 288. $3B+12C+18D=288$.

Subtracting 3B 1C 6D aequalia 80 from this last equation leaves,

11C 12D aequalia 208. $11C+12D=208$.

Subtracting 3B 4C 1D aequalia 54 from the same equation results in,

8C 17D aequalia 234. $8C+17D=234$.

The product of this equation and 11 is,

88C 187D aequalia 2574. $88C+187D=2574$

and multiplying 11C 12D aequalia 208 by 8, gives,

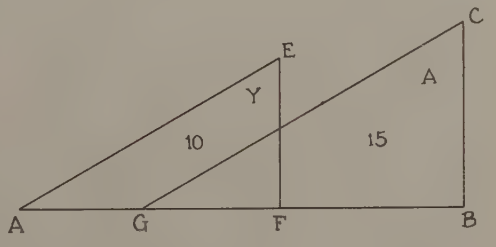
88C 96D aequalia 1664. $88C+96D=1664$

which subtracted from the one above leaves

91D aequalia 910. $91D=910$.

Simplifying, D = 10. After the proper substitutions Gosselin finds that C = 8, B = 4, and A = 6.

CEULEN, in his Arithmetische en Geometrische fondamenten published in 1615, uses simultaneous quadratic equations to solve some geometrical problems. In the accompanying figure the areas of triangles GCB and AEF are given as 15 and 10, respectively, with the further condition that there will be a common difference between the lengths of the segments desired. He, thereupon, wishes to find the lengths of AG, GF, and FB which is equal to EF, as well as the length of BC, giving the answer



$\surd.\surd 4784\frac{1}{16}-63\frac{3}{4}$ for AG, $\surd.\surd 321\frac{9}{16}-8\frac{3}{4}$ for GF, $\surd.\surd 59\frac{1}{16}\ 6\frac{1}{4}$ for FB, which is also equal to FE, and $\surd.\surd 1476\frac{9}{16}-18\frac{3}{4}$ for BC and $\surd.\surd 420-20$ for the common difference.

Ceulen then assigns $1\mathcal{U}$ (x) for AG; $1\mathcal{U}+1A(x+A)$ for FB; $1\mathcal{U}+2A$ for FB; and $1\mathcal{U}+3A$ for BC. Then GB, the base of the triangle is $2\mathcal{U}+3A$, and the product of GB and BC is $2\mathcal{Z}+9\mathcal{U}A+9AA$. The equation of the area of triangle GBC is $2\mathcal{Z}+9\mathcal{U}A+9AA$ gelijc 30. In the same manner he has a similar equation for the area of triangle AFE as $2\mathcal{Z}+5\mathcal{U}A+2AA=20$.

DE

Arithmetiſche en Geometriſche
fondamenten,

VAN

M^r. LVDOLF
VAN CEVLEN.

Met het ghebruyck van dien

In veele verſcheydene conſtighe queſtien, ſoo Geometrice door linien, als Arithmetice door irrationale ghetallen, oock door den regel Coſs, ende de tafelen ſinuum ghefolveert.

TOT LEYDEN,
By IOOST van COLSTER, ende IACOB MARCVS.
Anno cIↃ IↃ cXV.

Fig. 28. Title page of Ludolf Van Ceulen, De Arithmetische en Geometrische fondamenten (Leyden, 1615).

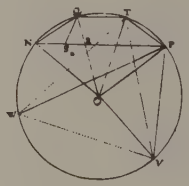
ALGEBRA
Ofte
STEL-KONST,

Beſchreven

Tot dienſt van de Leerlinghen,

Door

GERARD KINCKHUYSEN.



Tot HAERLEM,

By *Paſſchier van Wesbuſch*, Boeck-verkooper op de Marckt, in den beſlaghen Bybel. ANNO 1663.

Fig. 29. Title page of Kinckhuysen, Algebra ofte Stel-Konst (Haerlem, 1663).

REGULA COS,
OF
ALGEBRA,

Zijnde de alder-konſtrijckſten Regel om het onbekende bekent te maken.

Ofte

Een hooge Onderwyſinge / waer in geleert wordt het Uytrecken der Wortelen / ſoo bezre men begeeren mach.

De Speties in { Surdiſche getallen. / Twee-namige getallen. / Coſſiſche getallen.

De Vergelijckingen van 2, 3, &c., &c. met Exempelen daer toe dienende.

DOOR

J.R. BRASSER, geadmitteert Lantmeeter tot HOORN.

Noch

Is hier by gheworght de Gromettia van Nicolaus Petri Daventrienſis, ende andere Queſtien van de Algebræ.

Als mede

Eenige Exempelen van Gerrit Everts. Backer, Schoolmeeſter tot Gracht.

t'AMSTERDAM,

By Gerrit van Goedesbergh, Boeckverkoopr op 't Water / aen de Stuerwerugh / in de Delfſche Bybel. Anno 1663.

Fig. 30. Title page of Brasser, Regula Cos, of Algebra (Amsterdam, 1663).

ARITHMETICA.
Dat is
DE REKEN-KONSTE,
Beſchreven door

ANTHONI SMYTERS.

Waer inne geleert worden allerleye Rechteningen den koopmandel noodigh / ghebruckt in veel landen.

Het eerſte deel.

Inhoudende de ſoorten / te weten:

De Speties,
De Reductie van ghelde,
De Regel van drien,
Ende de Practijcke: ſoo wel in ghecheele als gebroken ghetalen.

Gecorr. geert ende vermeerdert door den Author ſelve na den ſeſte editie.

Ende ſijn deſe 4 Deelen van de Algebræ ende den Regel Cos in een Formaet op nieuws Gedruckt in 4°., om in een Bandt te vinden, voor den Leerlingh ende die 't behoeft.

t'Amſterdam, by Jacob van Leeſt, Boeck-verkooper op 't [illegible] Bybel. 1663.

Fig. 31. Title page of Smyters, Arithmetica (Amsterdam, 1663).

Expressing these simultaneously the result is,

9 $2z + 5p A + 2AA$ gelijc 20	$2x^2 + 5xy + 2y^2 = 20$
5 $2z + 9p A + 9AA$ gelijc 30	$2x^2 + 9xy + 9y^2 = 30$
$18z + 45p A + 18AA$ gelijc 180	$18x^2 + 45xy + 18y^2 = 180$
$10z + 45p A + 45AA$ gelijc 150	$10x^2 + 45xy + 45y^2 = 150$
$8z - 27AA$ gelijc 30 dat is	$8x^2 - 27y^2 = 30$

$$1A \text{ gelijc } \sqrt{\cdot \frac{8z - 30}{27}}$$

or $y = \sqrt{\frac{8x^2 - 30}{27}}$. It will be observed here that 9 and 5 before the equations are the multipliers of those equations for the purpose of elimination. Furthermore, the author does not indicate the negative root. He now gives the segments: GF as $1u + \sqrt{.}\ 8 - 30$, FB as $1u + \sqrt{.}\ 32z - 120$ and the fourth as $1u + \sqrt{.}\ 2\frac{2}{3} - 10$. Why Ceulen did not consider the denominator in these values is not explained, although he accounts for it in the remaining solution, when he finds that GF is $2u + \sqrt{.}\ 2\frac{2}{3} \div 10$ $(2x + \sqrt{2\frac{2}{3}x - 10})$. Here we notice the introduction of another symbol for minus, $\div$. In the same manner the other base is determined. By finding algebraically the areas of the triangles, he obtains the value of x, and consequently the required lengths of the segments become

$$AG = \sqrt[4]{4784\tfrac{1}{16} - 63\tfrac{3}{4}}$$

$$GF = \sqrt[4]{321\tfrac{9}{16} - 8\tfrac{3}{4}}$$

$$FB = FE = \sqrt[4]{59\tfrac{1}{16} - 6\tfrac{1}{4}}$$

$$BC = \sqrt[4]{1476\tfrac{9}{16} - 18\tfrac{3}{4}}.$$

The following method of solving equations of two or more unknowns is found in RAHN'S Teutsche Algebra (p. 81), which belongs to the transitional period.

The problem reads,

Three have money. The first -100 has as many as $\frac{1}{4}$ of the other two; the other -100 has $\frac{1}{3}$ of the other two; the third -100 has $\frac{1}{2}$ of the other two.

a = ?	1	$a - 100 = \frac{b + c}{4}$
b = ?	2	$b - 100 = \frac{a + c}{3}$
c = ?	3	$c - 100 = \frac{a + b}{2}$
$1 * 4$	4	$4a - 400 = b + c$
$2 * 3$	5	$3b - 300 = a + c$
$3 * 2$	6	$2c - 200 = a + b$
$4 \pm$	7	$4a - b - c = 400$
$5 \pm$	8	$-a + 3b - c = 300$
$6 \pm$	9	$-a - b + 2c = 200$
$7 + 8 + 9$	10	$2a + b = 900$
$7 - 8$	11	$5a - 4b = 100$
$10 * 4$	12	$8a + 4b = 3600$
$11 + 12$	13	$13a = 3700$
$13 \div 13$	14	$A = \frac{3700}{13}$ = der summ des ersten
$14 * 2$	15	$2A = \frac{7400}{13}$
$10 - 15$	16	$B = \frac{4300}{13}$ = der summ des andern
$14 + 16$	17	$A + B = \frac{8000}{13}$
$9 + 17$	18	$2C = \frac{10600}{13}$
$18 \div 2$	19	$C = \frac{5300}{13}$ = der summ des dritten

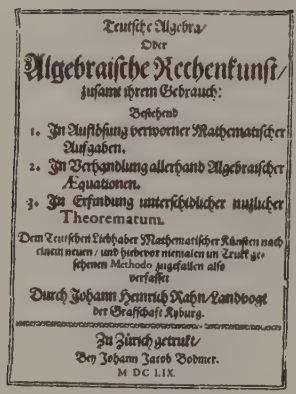
DESCARTES' reference to simultaneous equations is made in the statement[140]

Then, making no distinction between known and unknown lines, we must unravel the difficulty in any way that shows most naturally the relations between these lines, until we find it possible to express a single quantity in two ways. This will constitute an equation, since the terms of one of these two expressions are together equal to the terms of the other.

We must find as many such equations as there are supposed to be unknown lines; but if, after considering everything involved, so many cannot be found, it is evident that the question is not entirely determined. In such a case we choose arbitrarily lines of known length for each unknown line to which there corresponds no equation.

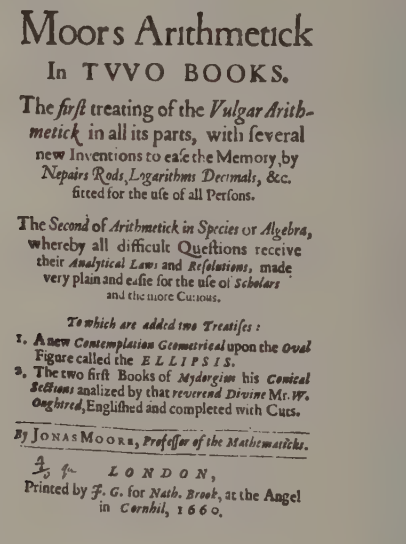
If there are several equations, we must use each in order; either in considering it alone or comparing it with the others, so as to obtain a value for each of the unknown lines; and so we must combine them until there remains a single unknown line which is equal to some known line, or whose

[140] Smith and Latham, The Geometry of René Descartes, 1925, p. 6.



Teutſche Algebra/
Oder
Algebraiſche Rechenkunſt/
zuſamt ihrem Gebrauch:
Beſtehend
1. In Auflöſung vorkommer Mathematiſcher Aufgaben.
2. In Verhandlung allerhand Algebraiſcher Æquationen.
3. In Erfindung unterſchidlicher nutzlicher Theorematum.
Dem Teutſchen Liebhaber Mathematiſcher Künſten nach einem neuen / und hiebevor niemalen im Truk geſehenen Methodo zugefallen alſo verfaſſet
Durch Johann Heinrich Rahn/ Landvogt der Grafſchaft Kyburg.
Zu Zürich getrukt/
Bey Johann Jacob Bodmer.
M DC LIX.

Fig. 32. Title page of Rahn's Teutsche Algebra (Zurich, 1659).



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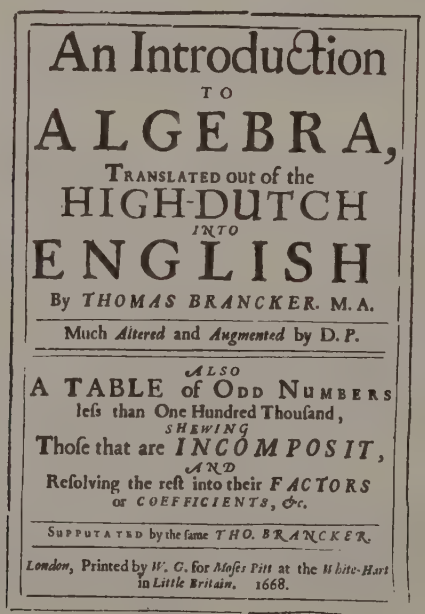
By JONAS MOORE, *Profeſſor of the Mathematicks.*
LONDON,
Printed by *J. G.* for *Nath. Brook*, at the Angel in *Cornhil*, 1660.

Fig. 33. Title page of Moors Arithmetick In Two Books (London, 1660).



An Introduction
TO
ALGEBRA,
TRANSLATED out of the
HIGH-DUTCH
INTO
ENGLISH
By *THOMAS BRANCKER*. M. A.
Much *Altered* and *Augmented* by D. P.
ALSO
A TABLE of ODD NUMBERS
leſs than One Hundred Thouſand,
SHEWING
Thoſe that are *INCOMPOSIT*,
AND
Reſolving the reſt into their *FACTORS*
or *COEFFICIENTS*, &c.
SUPPUTATED by the ſame *THO. BRANCKER.*
London, Printed by *W. G.* for *Moſes Pitt* at the *White-Hart* in *Little Britain*. 1668.

Fig. 34. A title page of Rahn's Teutsche Algebra translated out of the High-Dutch into English by Thomas Brancker (London, 1668).

square, cube,.....is equal to the sum or difference of two or more quantities, one of which is known, while the others consist of mean proportionals between unity and this square.....multiplied by other known lines. I may express this as follows:

$$z = b$$

$$\text{or } z^2 = -az + b^2 \ldots$$

It is noteworthy that the labor employed in treating this type of equation has gradually decreased. By the time of MOORE it has been reduced apparently to a minimum.[141]

To find three numbers, viz. A, E, Ƈ, with this condition that $A + \frac{2}{3}E + \frac{2}{3}Ƈ = z(100)$

$$E + \tfrac{3}{4}A + \tfrac{3}{4}Ƈ = z \text{ and } Ƈ + \tfrac{4}{5}A + \tfrac{4}{5}E = z$$

(1) $3A + 2E + 2Ƈ = 3z$
(2) $3A + 4E + 3Ƈ = 4z$ Reduced into Integers
(3) $4A + 4E + 5Ƈ = 5z$

By sub. (2, e, 3) it is $A + 2Ƈ = z$ and $Ƈ = \frac{z - A}{2}$.

By sub. (2, e, 1 and 2) it is 2A 2E = 2z and E = z - A.

By sub. (1, e, 2) it is $2E + 10 = z$, therefore $4z - 4A + z - A = 2z$ and $\frac{5z}{5} = A = 60.E$ 40.Ƈ 20.

c. QUADRATIC EQUATIONS:

The study of the solution of quadratic equations is introduced by making a comparative study of the rules for their reduction. In general, the rule is an adaptation of the Arabic method.

SCHEUBEL (1551), after stating the first canon, goes on to say of the coefficient of the second degree term that[142]

.....if it is unity then to the square of the number of the middle character, the number of the smallest character must be added; then from the square root of this sum, half of the middle character must be subtracted. This being done there will be some central number sought since the value of the root is expressed through what is left.

Marco Aurel (1552) and Salignacus (1580) give a similar rule.

DIGGES (1594) says,[143]

The Reduction which is sought in AEquation is to bring one part of the AEquation, to one simple Cossicall number, viz. 1⌿, 1⌿⌿, etc. which is done two ways.

First, by transposing or removing of numbers from one part to the other.

Then by reducing those numbers so transposed, to their least Denomination, or if they be Fractions, to an Integer.

In Chapter IX of Stratioticos, Digges explains "How to extract the second Radix or Quadrate roote of any Cossicall numbers, arising in any AEquations."[144]

Forasmuch as alreadie in Reduction, I have taught so to order the AEquation, that the Characters of the second part of the AEquation shal alway be lease than the singled Character. This is most Certaine, that the extraction of the second Roote, or square Radix of Cossicall Numbers, is never in any Arithmetical operation necessarie, save onely of these varieties following.

To finde the Quadrate or second Radix of
- A Number alone.
- ⌿ or Rootes onely.
- ⌿ + Numbers.
- Numbers - ⌿.
- ⌿ - Numbers.

These are the only five varieties, wherein the extraction of Rootes Quadrate in these Cossicall AEquations are required, and for every of these briefe Rules shall ensue.

Rules

1. For the first, I referre you onely to the first Booke of my Fathers, where you are taught out of my abstract or simple number or Fraction, to extract his Square Radix.

2. In the second, the number itself that is with the rootes coupled, is the number or Radix desired.

3. In the third, you shall take the moitie of the numbers of ⌿ or Rootes.

[141] Arithmetick, 1660, p. 93.
[142] Algebra, p. 19(v).
[143] Stratioticos, p. 45.
[144] Ibid,, p. 49.

This moytie square, adde the same to the Number Abstract in your AEquation.

To the Quadrate Radix of that Product, adioyne the moitie first vsed.

The Number resulting of this last Addition, is your desired Radix.

4. In the fourth, set downe (as before) the moitie of the + number that N. Square. That Square adde to the number Abstract, and from the Roote Quadrate of the Product, deduct the moytie first set downe. This Remaine is the desired Roote.

5. In the fifth kind there is alway two Rootes vnless the moytie of the + squared, be equal to the number Abstract, for then is the moytie of the number the Radix sought. But howsoever that fall out, you shal as before reserve the moytie of the Primes, first squaring the same, and from that Square, deducting the Abstract Number. The Roote Quadrate of the Remainder, added to your reserved moytie, maketh the greater Radix, the same quadrate roote deducted from the moytie reserved, leaveth the lesser Radix.

W. P(HILIPS) in his Pathway to Knowledge, 1596, page 338, gives several rules of reduction of which two are cited here: The second Equation or algeber's rule.

When 3 quantities are in naturall places, and that the 2 greatest are like the smallest, then devide the 2 least, by the greatest, and then multiply the half of the middlemost quantitie in it selfe, and add to the production the smalest quantity, and out of that production, extract √ and out of that take the halfe of the middlemost quantitie, and that which resteth is the worth of a 2ℓ.

The third Equation or rule of algeber.

When there are 3 quantities in their naturall order placed, and that the greatest and the smalest are like the midlemost, then devide the middlemost and the smallest by the greatest, and then multiplie the halfe of the middlemost quantity by it selfe, and out of that production, take the smalest quantitie, and out of the rest take the √, and to the production of this extraction, add the halfe of the middlemost quantitie, or take it out of the half of the middlemost quantitie as you shall see occasion, and the remainder shalbe the value of 1ℓ.

The fourth rule pertains to the type $ax^2 = bx + N$.

The directions of Henrion, Sesen, and Follinus correspond exactly to those of Clavius who, in general, employs the method of his predecessors. Clavius, together with Henrion and Sesen, divides the rule into four parts, namely, aequation, reduction, division, and quotient. Follinus, however, emphasizes the parts of the rule by asking five questions.[145]

1. Quid aequatio sit?
2. Quomodo reducatur?
3. Quid divisio?
4. Quid Extractio radicis?
5. Quotuplex existat?

It is of interest to note that the principal parts of the rule as formulated by Clavius, Follinus, Sesen, and Peucer resemble very closely those made by STIFEL, who gives them as

Inuentio
Divisio
Reductio
R. extractio

OUGHTRED, whose example is followed by Moore, also divides his rule into five parts:[146]

1. Reduction
2. Transposition
3. Application
4. Depression
5. Potestas

STEVIN outlines his rule by attacking a problem under the following headings:[147]

1. Explication du donne
2. Explication du requis
3. Construction
4. Demonstration Arithmetique
5. Autre domonstration Geometrique

Petri gives the rule, which varies little from any of those already cited, except in the conciseness of statement. Stampioen exhibits the influence of Petri in his Generalen Regel and comes to the principal parts of the rule without any unnecessary preliminary material.

[145] Algebra, 1622, p. 179.
[146] The Key of the Mathematics, 1647, p. 54.
[147] OEuvres de S. Stevin, 1634, p. 81.

De BILLY in the Regle d'Algebre directs that[148]

> You must first for the number unknown put 1R, and after, examine this root according to the tenour of the question, until you come to an Equation. Secondly, this equation is to be reduced, if need require. Thirdly, you must divide every part of the Equation by the number of the greatest cossick character. After which, either the quotient, or some root of the quotient, will give the root unknown. This is the rule of Algebra...

Herigon, in turn, introduces terms explained in an earlier section, when he suggests the reduction of equations by parabolismus first, and then adds that further reduction can be made by isomerie, hypobibasmus and antithesis. It will be recalled that Renaldini uses the same terms to describe his method of reduction.

PEUCER'S rule may be subdivided into

root extraction
division
equation
reduction

RAHN (1659) calls this rule "Casting away the Second Term," which he explains thus:[149]

> By the power of the first term divide the unknown quantity of the Second Term. Then if the Signs of the First and Second Terms be unlike, subtract the Quotient out of the Affirmative Root; but if the Signs be like, then add the same Quotient to the Affirmative Root. Then rectifie the Substitution, and you have your Desire.

No such rule was found in Feliciano, Gemma, Ghaligai, Diez, or Ceulen. Those who explain the rule with each type of equation are Nuñez, Buteo, Perez de Moya, Bombelli, Visconti, Rami, Unicorno, Cataldi, Graffenried, and Foster. Grisio uses Bombelli's method. The remaining authors, such as Peletier, Balam, Hume, Lauremberg, Wilken, Lantz, van der Huips, Descartes, and Gibson give a rule which does not differ essentially from those already cited.

The following problems are considered as typical illustrations of the rules of reduction of the quadratic equations discussed above, and are typical of their authors. One problem at least has been cited for each author except in the cases where an author, as for example Follinus, has incorporated bodily, material used by his predecessors. A great number of problems is quoted for the purpose of furnishing the reader with illustrations from each author cited. The illustrations are given as far as possible in chronological order up to 1630. Many of the authors after 1630 were influenced by analytical geometry to such an extent that the treatment of the equation underwent a complete change. Rahn, Harriot, Balam, Vieta, Descartes, Renaldini, Peucer, Brasser, Moore, Kinckhuysen and Twysden belong to this transitional period and will be discussed in a separate section.

FELICIANO in the edition of 1550 has left us this problem. His statement of the problem with the accompanying marginal solution has been reproduced exactly as it is found in the text.[150] (Please look on page 56 for this problem.)

Peletier's 1692 edition is a great improvement on the earlier editions. The style is similar to that illustrated on the following page; but the explanation is clearer and the marginal notes are somewhat condensed.

SCHEUBEL (1551) begins the section on algebra in his Euclid with the explanation of the symbols employed by Stifel; but, unlike Stifel, he retains the symbol for the constant term, namely, N for numerus. He goes on to explain that the unknown will be represented by Ra. for radix. Since the squared term is Ra. multiplied by itself, it will be designated by Pri. for primus, the first multiplication, and so on. This scheme gives rise to the oddity of designating x^n by the number n - 1, which is a scheme contrary to that used by Paciolo; for Tropfke shows that Paciolo write $R.30^a$ for the twenty-ninth power.[151]

Scheubel, after an explanation of the three types of quadratic equations, follows with a problem illustrating each type.[152]

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[148] Abrege Des Preceptes D'Algebre, 1637, p. 10. See LXIII in bibliography.
[149] Deutsche Algebra, p. 63. See XIV in bibliography.
[150] Libro di Arithmetica, p. $I^{(r)}$. [151] Cajori, Notations, p. 108. [152] Algebrae, 1550, p. $19^{(r)}$.

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There are three persons who formed a company. The first put in a certain amount. The second put in twice as much as the first plus 4 and the third put in the product of the first and second, that is to say, the third one put in as much as it may make by multiplying that which the first put in and that which the second put in. They found that they had earned 6968 ducats. The first received 80 ducats of the earning. Determine how much each put in. In order to do this, do it this way. Let the first one put in 1 cosa; the second put in 1 cosa plus 4; the third put in 2 censi plus 4 cose because 2 cose plus 4 by 1 cosa make 2 censi plus 4 cose, the amount put in by the third. If the total put in by the entire company is 2 censi plus 7 cose plus 4, then you should say for the rule, if 2 censi plus 7 cose plus 4 gives one 6968 ducats, what would give me one cosa, which the first one put in. Multiplying 1 cosa by 6968 makes 6968 cose. This divided by 2 ce. plus 7 co. plus 4 is equal to 80, which is the amount he received. Multiplying 80 by 2 ce. plus 7 co. plus 4 results in 160 ce. plus 560 co. plus 320. Subtracting the cosa-terms, that is, 560 cose from 6968 cose leaves 6408 cose. Now you have the principle which says, that when the censi and the number are equal to all the cose, the cosa divides itself into half and this half is, then, multiplied by itself. That which is the product is subtracted from the number, and the root of the remainder in question added to one half the cosa which had been used, equal the value of the cosa: first, reduce the equation to one censo by dividing by the (coefficient) of the censi; but dividing 320 by 160 will give 2, and dividing 6408 by 160 will give $40\frac{1}{20}$ cose, a mixed fraction. Now you have $x^2 + 2$ equal to $40\frac{1}{20}x$. Then taking one half of $40\frac{1}{20}$ gives $20\frac{1}{20}$ multiplied by $20\frac{1}{20}$ makes $\frac{641601}{1600}$ which is $401\frac{1}{1600}$. If 2 is taken from this, $399\frac{1}{1600}$ results and the square root of $399\frac{1}{1600}$ is $19\frac{39}{40}$, the quantity, which added to the half of cosa, $20\frac{1}{40}$, makes 40, the value of cosa. That much plus 4 is the amount the first put in. The third put in 3360, that is, 40 by 84, etc. In order to know how much the second and the third received, do as you did for the first. You will find that the second received 68 ducats and the third received 67 ducats and in the same way you will know the rest.

The substance of this is as follows in modern notation:

x = amount the first had.
$2x + 4$ = amount the second had.
$2x^2 + 4x$ = amount the third had.
$2x^2 + 7x + 4$ = amount they all had
$2x^2 + 7x + 4 = 6968$

$$\frac{6968x}{2x^2 + 7x + 4} = 80$$

$$6968x = 160x^2 + 560x + 320$$

$$\begin{array}{r} 6968x \\ 560x \\ \hline 6408x \end{array}$$

$$160x^2 + 320 = 6408x$$

Dividing through by 160 gives $1x^2 + 2 = 40\frac{1}{20}x$. The first is 40. The second is 84. The third is 3360.

Canonis Primi.	First Rule.
1 pri. + 8 ra. 65N	$x^2 + 8x = 65$
4 in se 16 + 65	$\frac{1}{2}$ of 8 is 4
veniunt 81. Huius radix,	$4^2 = 16$
sunt 9. minus 4,	$16 + 65 = 81$
manet 5.	$\sqrt{81} = 9$
Atq tantus est radicis valor.	$9 - 4 = 5$, the value of the root

Exemplum Canonis Tertii.	Example of the Third Rule.
1 pri. + 12N, aequalis 8 ra	$x^2 + 12 = 8x$
4 in se 16 minus 12	$\frac{1}{2}$ of 8 = 4
manet 4. Huius radix quadrata	$4^2 = 16$
sunt 2 {de ad^4 & manet 2,	
vel	$16 - 12 = 4$
proueniunt 6.	$\sqrt{4} = 2$
Vterque radicis valor, ac probationi conueniens numerus.	2 {from to 4 makes 2 or 6, the value of the root.

It will be observed in the examples cited above that in the first one the equality sign was employed, but the word for equality was used in the other. Here, too, Scheubel seems to avoid negative roots; for in the first equation, $x^2 + 8x = 65$, the roots are -13, 5 and he gives only 5. The same is true of other problems resulting in negative roots. In the second example cited, both roots are given, since both are positive. The solution of some of the problems is arranged in the following manner:[153]

Pri.	ra.	N	
4	+ 3	aequales 217	$4x^2 + 3x = 217$.

He explains that, because the "maximi characteris numerus" is not unity, one must divide through by 4, in which case he gives

Pri.	ra.	N	
1	+ $\frac{3}{4}$	aequales $\frac{217}{4}$	$x^2 + \frac{3}{4}x = \frac{217}{4}$.

Proceeding as in the earlier examples, Scheubel found the root to be 7, whereas the roots are 7, $-7\frac{3}{4}$.

The explanation of the three basic rules is followed by their applications. Problem 5 on page 22 reads thus:[154]

Propositum est dividere numerum 8 in duas partes, quarum secundae quantitates, vnà cum primis, & his ipsis numeris, 194 (original had 199) faciant, quaeritur, etc.	It is proposed to divide the number 8 into two parts, whose cubes, together with their squares and the number itself makes 194; it is desired, etc.
Facit 5 & 3	Answer: 5 and 3

Operatio		Operation	
1 ra.	8N - 1 ra.	x	8 - x
1 pri.	64N - 16ra + 1 pri.	x^2	$64 - 16x + x^2$
1 se.	512N - 192ra + 24 pri. - 1 se.	x^3	$512 - 192x + 24x^2 - x^3$

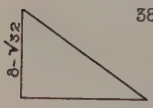
584N - 208 ra. + 26 pri. aequales 194N.	$584 - 208x + 26x^2 = 194$
Vel additis & subtractis aequalibus,	or adding and subtracting equals we get
veniunt 26 pri. + 390N aequales 208 ra. Est autem exemplum canonis tertij, atque radicis valor 3 vel $\overline{5}$, vt lubet, prior pars.	$26x^2 + 390 = 208$. This is an example of the third rule; and therefore the value of the radix is 3 or 5.

The next example illustrates one of the geometrical applications given by Scheubel.[155]

Primum. Esto triangulum rectangulum, atq cathetus eius $8 - \sqrt{32}$, basis verò & hypotenusa simul, $16 - \sqrt{128}$, quanta erit vtraque, basis scilicet & hypotenusa linea seorsim quaeritur.	What will be the base and hypotenuse if the vertical side (cathetus) is $8 - \sqrt{32}$ and the hypotenuse and base together are $16 - \sqrt{128}$?
Basis quidem $6 - \sqrt{18}$ Hypotenusa verò $10 - \sqrt{50}$	Answer: Base, $6 - \sqrt{18}$ Hypotenuse, $10 - \sqrt{50}$

Operatio	Solution
Cathetus ex hypothesi, sunt $8 - \sqrt{32}$	Side, $8 - \sqrt{32}$

[153] Algebrae, 1550, p. 20(r).
[154] Ibid., p. 22(v).
[155] Ibid., p. 39.(r).

Sit autem nunc basis 1 ra Hypothenusa igitur, erunt 16 - √128N minus 1 ra.	Base, x Hypotenuse, $16 - \sqrt{128} - x$
 384 - √131072N, plus 1 pri. minus 32 - √512ra.	$384 - \sqrt{131072}\ x^2 - 32 - 512x$ is the square of the hypotenuse
384 - 131072N minus 32 - √512ra. aequales 96 - 8192N	$384 - 131072 - 32 - \sqrt{512x} = 96 - 8192$
288 - 73728N aequa. 52 - √512ra.	$288 - 73728 = 52 - \sqrt{512x}$
exeunt 6 + √18 & tanta est basis quantitas.	$x = 6 + \sqrt{18}$.
Et quia iam basis quantitas nota est, quanta hypotenusa sola fuerit, cum hae duae quantitates simul, ex hypothesi, 16 - √128 sint, subtractione manifestabitur.	Subtracting the value of x from the sum of the base and hypotenuse, will give the hypotenuse, which is $16 - \sqrt{128}$.

In this problem all the explanatory paragraphs have been omitted.

The source of the examples cited above is Scheubel's Algebrae compendiosa facilisque descriptio, which is a reprint of the algebraic part of Scheubel's Euclid, 1550. "It is of importance as representing the first appearance in France of the symbols + and - and of some other German symbols in Algebra."[156]

CHARLES HUTTON praises the Algebrae compendiosa (1551):[157]

> The work is most beautifully printed and is a very clear though succinct treatise; and both in the form and matter much resembles a modern printed book.

Peter Ramus says that when traveling through the European countries he called upon all the great mathematicians, among whom he preferred the Germans as a whole, and that Scheubel should be honored above all his contemporaries as the pride of Tübingen.

MARCO AUREL, who was a German residing in Spain, tells us that his book is the first algebra published in Spain. In this he states a problem for which he gives three different methods of solution. The second solution is a pure biquadratic but will be inserted here for the purpose of comparison with the other solutions. The problem is,[158]

Da me vn numero que multiplicado en si mesmo, venga 36.	Give me a number which multiplied by itself becomes 36.
Let the number be 3 ℛ ; multiplied into itself, gives 9 ʒ, equal to 36 ℚ: divide 36 ℚ by 9 ʒ, and 1 ℛ will have the value of 4 ℚ: whose square root is 2 ℚ. This is the value of one ℛ , and you have 3 ℛ , whose value is 6 ℚ, which is the number desired, as shown in the second equation.	$3x$ = the number $3x \cdot 3x = 9x^2$ $9x^2 = 36$ $x^2 = 4$ $x = 2$ $3x = 6$ or the number

It will be observed that negative roots are disregarded.

Again: Given a number which multiplied by itself gives 36.

Take the "numero q̄" to be 2 ʒ: multiplying into itself, gives 4 ʒʒ equal to 36 ℚ. Dividing 36 ℚ, by 4 ʒʒ, gives 1 ʒʒ the value of 9 ℚ. If you take √ of 9, it will be the value of one ℛ , q̄ is 3; the value of ʒ is 3, 2 ʒ will have the value of 6. This is the number desired. Multiply it by itself it will be 36, and so on.	$2x^2$ = the number $2x^2 \cdot 2x^2 = 4$ $4x^4 = 36$ $x^4 = 9$ $x = \sqrt[4]{9}$ or $\sqrt{3}$ $2x^2 = 6$ $(2x^2)^2 = 36$

In the third solution of this problem he lets the unknown quantity be $2\mathcal{R} - 2$; multiplying this by itself

gives 4 ʒ - 8 ℛ + 4 ℚ equal to 36 ℚ;....and the value	$2x - 2$ = the number $(2x - 2)^2 = 4x^2 - 8x + 4$

[156] Cajori, Notations, p. 150.

[157] Idem.

[158] Arithmetica, 1552, p. 76r.

of one ℛ will be 4𝒬; but you have taken 2 ℛ - 2𝒬. The 2 ℛ has the value 8, from which quantity subtract 2 and it will be 6.

$4x^2 - 8x + 4 = 36$
$x = 4$
$2x - 2 = 6$

These problems, which exhibit Arabic influence, are followed by an explanation of each of the eight types mentioned earlier in this treatise. Since this section deals with the quadratic equation only, the fifth, sixth, and seventh rules will be considered here.

In the fifth rule, where "the two greater are equal to the lesser," AUREL gives the directions for the solution of $ax^2 + bx = N$ as follows:[159]

Partiras la menor, y mediana cada vna por si por la mayor, y multiplicaras la metad del quociente, dela particion del mediano en si mesmo: y al dicho producto juntaras el quociente dela particion del menor: √ de toda esta summa: menos la ½ del quociente del mediano, sera la valor de vna ℛ .

It will be observed that here again he follows the Arabic custom.

Exemplo. 2𝒥 + 12 ℛ son ygual a 32𝒬;... He gives the solution rhetorically, the steps of which are indicated below:

$2x^2 + 12x = 32$
$x^2 + 6x = 16$

½ of 6 is 3, which squared is 9. Adding this to 16 gives 25.

$x = \sqrt{25} - 3$
$x = 5 - 3$ or 2.

In the sixth rule the process explained by the fifth rule is followed in the equation "2𝒥 + 32𝒬 son ygual a 20ℛ," or $2x^2 + 32 = 20x$. After dividing through by two and taking one half of the coefficient of x, or 5, he squares the 5, but, instead of adding it to 16, he obtains the difference between 25 and 16, namely 9.[160]

.....delos quales quita 16, que es el quociente del menor, y quedaran 9: √9 - 5. Digo 5 - √9, que es 2, sera la valor de vna ℛ .

He goes on to say that √9 + 5, or 8, is also a value of 1ℛ. When x has the value of 2, $2x^2$ will be 8, and 8 plus 32 make 40, which is equal to the other value of 20x. If x is 8, $2x^2$ is 128. Adding 32 to this will make 160. The other quantity also has the value of 160. Aurel then explains that √9 - 5 cannot be used, for it is impossible to take 5 from 3.

The solution under the seventh rule, which treats with another of the quadratic types, is straightforward and applies the method given in rule five.

Marco Aurel gives (p. 80r) the following method of simplification. When 6 ℛ + 4 𝒬 are equal to 46𝒬, subtracting 4 𝒬 from each side leaves 6 ℛ equal to 42𝒬, (6x = 42); one ℛ will have the value of 7𝒬, (x = 7). Whenever negative quantities occur, the equation is reduced to one with all positive terms by adding the same quantity to both sides; and, consequently, it can be classed under one of the eight types given by Aurel.

Each subsequent chapter deals with problems illustrating one of the rules of reduction and, assuming an adequate knowledge of reduction, proceeds to the solution. For example, in Chapter XX, p. 121r,

Give two numbers, such that one will be 2 more than the other and the sum of their powers (squares) will be 394.

His solution can be reduced to these steps:

x = the lesser
$x + 2$ = the greater
$(x)^2 + (x + 2)^2 = 394$
$2x^2 + 4x = 394$
$x = 13$, or lesser
$x + 2 = 15$, or greater.

In another problem[161] he wishes to divide 10 into two parts such that 3 times the quotient of the greater and the lesser is equal to 27 times the reciprocal of the above quotient. He then cross multiplies and obtains 300 + 3𝒥 - 60 ℛ equal to 27𝒥, ($300 + 3x^2 - 60x = 27x^2$). The value of the lesser number thereupon becomes $2\frac{1}{2}$ and the larger $7\frac{1}{2}$, whose sum is 10.

[159] Arithmetica, 1552, p. 78r.
[160] Ibid., p. 79v.
[161] Ibid., p. 122r.

The problems on dividing a number into a required number of parts are followed by the "merchant" problems. These should prove valuable in the history of commerce. The Italian writers dealt, in general, with the commodity, cinnamon. AUREL cites such a problem on cinnamon, given by Paciolo,[162] in the following reference:

Fra Lucas del Burgo pone esta demanda en al 6° tratado, distincion 6ª, articulo 14°, question 21ª.

Since it is a problem in cubics, its solution will not be given here. However, in passing it may be well to note his attitude toward irrationals. He changes the numbers in Paciolo's problem so that the new ones will make the results rational, thus avoiding long and fatiguing labor:

pourque si vinieren irracionales fuera largo en escrivir, y muy fatigoso en hazer.

In the next chapter, however, there is no attempt at avoiding irrationals. The third problem, although simple, illustrates this point, as well as the fact that Aurel recognized both roots when they were positive.[163]

Hazed de diez tales dos partes, que multiplicando la vna con la otra vengan tres. Pongo que la vna parte sea vna ℒ, la otra sera 10ℚ - 1ℒ: multiplicado la vna con la otra, vienen 10ℚ - 1ℨ, ygual a 3ℚ. Ygual a dando a cada vna parte vn ℨ, y vernan 10ℒ, ygual a 1ℨ + 3ℚ, verna vna ℒ a valer 5 + √22, que es la vna parte; la otra 5 - √22. Las dos juntas hazen 10: y multiplicado la vna con la otra vernan 3.

Divide a number into two parts such that one multiplied by the other will give 3.

x = 1 part
$10 - x$ = the other
$10x - x^2 = 3$
$x = 5 + \sqrt{22}$
$x = 5 - \sqrt{22}$

The last two chapters give many problems on cubics and biquadratics.

J. REY PASTOR, in Los Matematicos Españoles Del Siglo XVI, 1926 (p. 103), states that the book of Aurel exerted a great influence in the development of mathematics in Spain.

GHALIGAI, who follows the order of equations peculiar to Al-Khowarizmi, gives nineteen rules in order to explain his classification of equations. For the first example (§155) he states that if things (cose) are equal to things, there must be, say, 12 things equal to 12 things and not 12 things equal to 11 things.

The next rule deals with a square equal to a thing, a square equal to a number, a thing equal to a number. This leads one to those rules which explain the types of affected quadratics.[164]

Qvando e □ & le cosa sono equale al numero, parti ogni cosa ne □ poi dimez za le cose, e una parte salua, & l'altra multiplica in se, e quel o che sa aggiuni al numero & la R della somma m° el dimezamento delle cose, tanto uale la cosa.

This is again the Arabic rule for reducing the equation $ax^2 + bx = N$ in which one half of the coefficient of x is taken, squared, and added to the number. One half of the coefficient of x is then subtracted from the square root of the sum, which gives the value of the unknown. The rules for considering the types $ax^2 + N = bx$ and $ax^2 = bx + N$, give a similar reduction, while the remaining rules are formulated for the cubic and biquadratic equations. The eleventh book contains problems based on the second book of Euclid. The next chapter is devoted to the problems of Leonard of Pisa. In this chapter GHALIGAI (1552) (fol. 98, incorrectly numbered) gives the problem shown on the next page. The reader's attention is called to two facts: the first is the use of--for sign of equality; the second is the imitation of the Arab custom of assigning a squared number to the unknown quantity. These evident Arabic influences

[162] Arithmetica, 1552, p. 125ʳ.
[163] Ibid., p. 127ʳ.
[164] Practica d' Arithmetica di Francesco Ghaligai Fiorentino, 1552, p. 90ʳ.

Truova un n°, che trattone la sua R̃ resti 72 domando, el detto n°, poni el detto n° sia 1□ che la suo R̃ per la 2 del 10, e 1 c° tratta di 1□ resta 1□ m° 1 c° questo è equale a 72 numeri, che sequendo l'ordine della 16l del 10, trouerrai ualere la c° 9 & p la 2 del 10, uale el □ 81. Adunque il detto n° su 81.

Find a number such that its root subtracted from it leaves 72. Let the number be x^2, whose root by section 2 of chapter 10 is x. This subtracted from x^2 leaves $x^2 - x$ equal to 72. Following the rule of section 16l of chapter 10, the value of x is 9 and by section 2 of chapter 10 the value of x^2 is 81.

1□	x^2
1 c°	x
1□ m° 1 c° --72 numeri.	$x^2 - x = 72$
$\frac{1}{2}$ $\frac{1}{4}$	$\frac{1}{2}$ $\frac{1}{4}$
$8\frac{1}{2}$ $72\frac{1}{4}$	$72\frac{1}{4}$
$\frac{1}{2}$ la R̃ $72\frac{1}{4}$	$\sqrt{72\frac{1}{4}}$ is $8\frac{1}{2}$
Fa 9 e $8\frac{1}{2}$	$8\frac{1}{2} + \frac{1}{2} = 9$, or value of x.
	$x^2 = 81$, the required number.

are to be expected from his familiarity with the work of Al-Khowarizmi.

JACQUES PELETIER du Mans resided in Paris, Bordeaux, Lyon and Rome. He died in Paris. His algebra shows in its symbols both Italian and German influence. Peletier speaks of Nuñez and Riese but questions whether their works were published. His admiration for Scheubel accounts for the extensive use he made of his works.[165]

Chapter XXV of the edition of 1554 is devoted to problems which require the extraction of roots. The first example gives two numbers in double proportion which, added together, give the same result of the product of the one by the other.

Il y a deus Nombres an proporcion Double: lequez ajoutez ansamble: sont autant, comme multipliez l'vn par l'autre.[166]

Ce sont 1R̃, e 2R̃. L'Addicion fèt 3R̃: la multiplicacion fèst 2ȿ. Donq 2ȿ sont egauz à 3R̃: E par diuision, 1ȿ èst egal a $1\frac{1}{2}$ R̃.

Maintenant, faut sauoèr qui ét la R̃ Çansique de $1\frac{1}{2}$R̃. Il n'y à' autre chose a fere, sinon oter le moindre nombre Cossique du plus grand: Sauoèr èst, oter 1R̃ de 1ȿ: demeurera 1R̃, egal a $1\frac{1}{2}$. Donq $1\frac{1}{2}$ èst le premier Nombre: L'autre sera 3 l'Addicion de 3 a $1\frac{1}{2}$, fèst $4\frac{1}{2}$: e la multiplicacion de 3 par $1\frac{1}{2}$, fèst aussi $4\frac{1}{2}$.

x = one number
2x = the other
3x = $2x^2$
x = $1\frac{1}{2}$, or first number,
2x = 3, or second number,
$1\frac{1}{2} + 3 = 3 \cdot 1\frac{1}{2}$
$4\frac{1}{2} = 4\frac{1}{2}$

Another problem which is of particular interest because it was used by writers subsequent to Peletier,[167] reads as follows:

Il y à vn Nombre: du Canse duquel, si vous otèz $\frac{1}{3}$ e $\frac{1}{4}$ e ancor' 8: e puis si vous multipliez le surplus por soeméme (sic) le produit sera egal au Çanse du Nombre que je di, joint a 13.

x = the square of the number
$x - \frac{1}{3}x - \frac{1}{4}x - 8 = \frac{5}{12}x - 8$
$(\frac{15}{2}x - 8)^2 = x + 13$

$\frac{25}{144}$ ȿ p.64 m.$6\frac{2}{3}$ R̃, egaux a 1R̃ p.13

$\frac{25}{144}$ ȿ sont egauz a $7\frac{2}{9}$ R̃ m.51,

a quoe est egal 1ȿ? Ce seront $\frac{1104}{25}$ R̃ m. $\frac{7344}{25}$. Auqueles est egal 1ȿ. Ce (nombre du Racines) sont $\frac{900}{25}$: c'est a diré 36 qui est le Çanse que nous voulions.

$\frac{25}{144}x^2 + 64 - 6\frac{2}{3}R = x + 13$
$\frac{25}{144}x^2 = 7\frac{2}{9}x - 51$
$x^2 = \frac{1104}{25}x - \frac{7344}{25}$
x = 36, the square of the number desired.

The term Çanse means square, for on p. 87, Peletier says, "Le Çanse de 1R, est 1ȿ." In another place he symbolizes Cansicanse by 1ȿȿ which is the fourth power of a number. The R Cansique evidently means the square root. The word "census" used by Gerard of Cremona for square became "zensus" in German; square of square was zensi-zens. When this was taken back into French the z became ç and so we have

[165] M. S. Day, Scheubel as an Algebraist, 1926, p. 22.
[166] L'Algebre, p. 79.
[167] Ibid., p. 88.

"çanse," and finally "canse."

Aside from the method of its solution, this problem is of some import because it is taken from Cardan who, as Peletier says, took it from "Mahommet Arabe," adding that

jè changè les nombres, e l'explicacion aussi: laquelle il fèst vn peu obscuremant: e la fèst tomber sus vn Nombre quarrè Irracionnal.

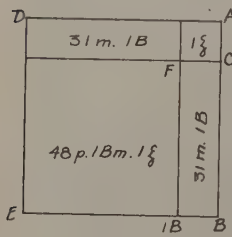
Though the two editions of Peletier examined do not differ materially, the French edition of 1564 includes some problems not found in the Latin edition of 1560. The problem quoted above is found only in the earlier edition. One problem that occurs in both editions is worthy of attention because of its geometrical application and because of Peletier's use of second roots. The problem is:[168]

There are two numbers, such that the difference of their squares is 48 and their sum added to their products gives 3. Find the two numbers.

Here the author interpolates the observation that, in order to use algebra perfectly, a knowledge of all kinds of theorems is presupposed, as of geometry, astronomy, music and physics:

e brief de tous ars e sciances (sic).

In order to illustrate his point he resorts to the fourth proposition of the second book of Euclid, who says that when the whole line is divided into two parts, its square is equal to the squares of the two parts joined to the product formed by multiplying one by the other.



The side AB is divided by the point C, and each of the two parts has a square: the other of these are two rectangles which are called supplements and which are obtained by multiplying the two parts AC and CB.

All of this makes up the whole square of the line AB.

Peletier asks the reader to recall that the product of two numbers results in the mean proportional between their squares, as 4 multiplied by 5 makes 20, which is the mean proportional between 16 and 25. According to this the supplements BF and DF constitute the mean proportional between the particular squares AF and EF. These added together will give the total square on the line AB.

Peletier then takes 1℥ (x) for the first number and 1A for the second. Their sum he designates by 1B; for, if the product is added to these two numbers, the result will be 3l. The square of 1℥ is ℥ (x^2) and that of A is 1A ℥ (A^2). Since 1B is taken as the sum of the two numbers sought, he subtracts it from their squares and obtains 48. The sum of their squares is then equal to 48 p. 1B. Since the first square is 1℥, the second is 48 p. 1B m. 1℥ and the supplement or the mean proportional will be 31 m. 1B.

Now the sum of the two squares that is 48 p. 1B, joined to twice the supplement, or 62 m. 2B, is equal to the total square of 1B, that is, to 1B ℥ (B^2). By transposition and reduction 1B ℥ is equal to 110 m. 1B ($B^2 = 110 - B$), from which 1B is found to be 10, the supplement 21, the two squares 49 and 9, giving 7 and 3 for the two numbers.

He adds a paragraph in which he seems to wish to convince his readers that "this question is beautiful: because from it result many beautiful questions. This is Stifel's, making only a change in the numbers."

LE RAMÉE (Ramus or Rami), who has been quoted by Graffenried, Rocha, Salignacus and Coutereels, has devoted only eleven pages of the edition of 1599 of his Arithmetica to equations and fewer still in his edition of 1613. His definitions indicate Greek influence. Without any further preliminary material he begins the section on algebra with the famous Alexander problem. He follows the regular method of solution. His symbolism and style of writing have been faithfully reproduced by Salignacus.

The method of presentation of the solution of problems by JUAN DIEZ FREYLE in his Summario Compendioso (1556), the earliest mathematical work published in America, resembles somewhat that of Ortega. In the part entitled Del arte Mayor, the

[168] L'Algebre, p. 113.

problem is stated and then followed by the "Regla," replacing Ortega's "Repuesta." Most of the problems are pure cubics; only the first two examples deal with pure quadratics. The second problem (p. 51) is

A man takes passage in a ship and asks the master what he has to pay. The master says that it will not be any more than for the others. When the passenger again asks how much it will be, the master replies: "It will be the number of pesos which, multiplied by itself and added to the number, will give 1260." Required to know how much the master asked.

Let the price be cosa of pesos. Then half of a cosa squared makes $\frac{1}{4}$ of a zenso, and this added to 1260 makes 1260 and a quarter, the root which, less $\frac{1}{2}$ of a cosa, is the required number. Reduce 1260 and $\frac{1}{4}$ to fourths; this is equal to $\frac{5041}{4}$, the root of which is 71 halves; subtract from it half of a cosa and there remain 70 halves, which is equal to 35 pesos, and this is what was asked for the passage.	x = price in pesos
	$x^2 + x = 1260$
	$\frac{1}{2}$ of 1 (coefficient of x) $= \frac{1}{2}$
	$1260 + \frac{1}{4} = 1260\frac{1}{4}$
	$1260\frac{1}{4} = \frac{5041}{4}$
	$\sqrt{\frac{5041}{4}} = \frac{71}{2}$
	$x = \frac{71}{2} - \frac{1}{2}$ or 35, the cost of passage.

Proof: Multiply 35 by itself and you have 1225; adding to it 35, you have 1260, the required number.

This is taken from D. E. SMITH'S translation of the Summario. However, the fact that Freyle really gives an erroneous solution is passed over without comment by the translator. The method of solution used so generally is of Arabic origin. The Arabs spoke of taking a half of the "thing" or quantity, meaning the coefficient of the quantity. In both problems cited Freyle says to take half of a cosa ($\frac{1}{2}x$), the square of which he calls "$\frac{1}{4}$ of a zenso" ($\frac{1}{4}x^2$). This, of course, is incorrect, although he actually completes the solution correctly by using $\frac{1}{4}$, which is the square of $\frac{1}{2}$ of the coefficient of the unknown quantity.

PEUCER, a professor of mathematics and medicine, devotes very little of his text to the equation. Only the last eight pages deal with the pure and affected quadratic equation. In one problem he wishes to find two numbers in triple proportion, which if they are multiplied one by the other, and their sum is subtracted from this product, the result will be 15.[169]

Sint inquirendi duo numeri in proportione tripla, quorum si ducas unum in alterum, de producto verò reijcias aggregatum utriusque, relinquantur 15. Sume pro minore 1ℵ, pro triplo eius 3ℵ. Multiplica unum in alterum, fiunt 3ʒ. A quibus si reijcias (sic) aggregatum radicum, aequantur 3ʒ - 4ℵ & 15. Et amplius 3ʒ & 15 + 4ℵ aequantur. Facta ergo diuisione 1ʒ ualet $\frac{15 + 4\aleph}{3}$. Ex hoc numero educendam esse radicem quadratam indicat character ʒ. Dimidium numeri radicum, id est, $\frac{2}{3}$ ducantur in se: & fiunt $\frac{4}{9}$. Quibus si $\frac{15}{3}$ adiungas, prodeunt $\frac{49}{9}$. Ex his radix quadrata ualet $\frac{7}{3}$. Cui si rursus addas dimidium numeri radicum $\frac{2}{3}$, fiunt $\frac{9}{3}$ uel 3 integra. Tantus fuit numerus minor. Triplum eius scilicet 9 maior est numerus. In quem si 3 ducas, fiunt 27. Inde si abijcias aggregatum numerorum inuentorum 12, restant 15, ut proponit quaestio.	x = the lesser number $3x$ = the greater number $3x^2 - 4x = 15$ $x^2 = \frac{15 + 4x}{3}$ $\frac{1}{2}$ of $\frac{4}{3} = \frac{2}{3}$ $(\frac{2}{3})^2 = \frac{4}{9}$ $\frac{15}{3} + \frac{4}{9} = \frac{49}{9}$ $\frac{49}{9} = \frac{7}{3}$ $x = \frac{2}{3} + \frac{7}{3}$ or 3, the first number. 9 is the second number.

FORCADEL gives very few problems on the quadratic equation, and those given are pure quadratics. In one problem[170] it is desired to find a number which multiplied by $\frac{2}{3}$ of itself will make 150. After giving the arithmetical solution he proceeds to the algebraical equivalent by representing the unknown by ra., $\frac{2}{3}$ of which is $\frac{2}{3}$ ra. Then 1 ra. by $\frac{2}{3}$ ra. gives $\frac{2}{3}$ P. because "1 premier nombre par 1 primer nombre," gives the second number. Thereupon

[169] Sectiones, 1556, p. V VI.

[170] Le Troysieme Livre de l' Arithmetique de P. Forcadel de Beziers, 1557, p. 71^{v}.

ARITHMETICA,

DAT IS:

REKEN-KONSTE,

Beschreven door

ANTHONI SMYTERS

HET DERDE DEEL.

Inhoudende de Fondamenten van de Regel Coss ofte Algebra.

t'AMSTERDAM,

Door Jacob van Leest, Woonende op 't Water/ by de Oude-brugh/ in de Blaeuwe kroon/ 1661.

Fig. 35. Title page of the third part of the 1661 edition of Smyters, Arithmetica (Amsterdam, 1661).

ALGEBRA OFTE REGEL COS,

Beschreven door

ANTHONI SMYTERS.

HET VIERDE DEEL.

Waer inne niet alleen lustighe, maer oock Konst ende nutte exempelen, met by-gevoeghde onderrichtingen, vergadert zijn.

t'AMSTERDAM,

Door Jacob van Leest, Woonende op 't Water/ by de Oude-brugh/ in de Blaeuwe kroon/ 1661.

Fig. 36. Title page of fourth part of the same text. Below are two pages of examples of quadratic equations found in this text.

33 Eener vraghe my hoe out ick ben/ den welcken ick antwoorde/ ick ben soo veel jaren out/ dat als ick 10 subtraheere van het vierde gerale mijnes ouderdoms/ ende oock subtraheere van mijnen ouderdom 35 jaren/ radix quadrata deser reste/ eveen soo veele is als de voorgheschrevene reste: hoe out ben ick?

Set 1 ✱ jaren, soo komen ½✱ − 10 ghelijck √ 1 ✱ − 35, multiplic. aen beyde zijden in sich komen.

Equatie.

1 ⊠ − 40 ✱ + 100 = 1 ✱ − 35 dat is/
1 ⊠ = 80 ✱ − 1600 = 16 ✱ − 560
de ⅟ 48
in sich 2304
Nota. trecht 2160
rest √ 144 dats 12 tot 48. Facit 60 jaren.

34 Eenen verkooper Peper/ de 4 ℔ om 8 ♄ min dan om soo vele 8 alsser 18 Pepers geweest zijn: Soo hy de ontfangene somme multiplicéert met 144/ soo bevint hy dan den radix cubica des products soo veel is/ alsser 18 Pepers verkocht zijn: Vraghe, hoe veel ℔?

4 ℔ 1 ✱ − 8 ♄ 1 ✱ ℔ komen ⅟ − 2 ✱ gemultip. met 144
komen √ ce 36 ⊠ − 288 ✱ gelijck 1 ✱
Equatie.
36 ✱ − 288 ✱ = 1
de ⅟ 18
in sich 324
trecht 288
den √ 36 is 6 getrocken van 18 rest facit 12 ℔ Peper.

35 Vint een gheral/ wanneer men 4 daer toe addeert/ daer na van het ghevonden gheral 6 subtraheert/ het gemeerdeert met het gemindert multiplicéert datter 200 komen.

Set 1 ✱ + 4 ende 1 ✱ − 6 t'samen gemultipl. maecken 1 ⊠ − 2 ✱ − 24 die zijn gelijck 200, dat is,

1 ⊠ = 2 ✱ + 224
de ⅟ 1 Nota 1 add.
in sich 1 225
den √ 15
add. 1
Facit 16

NOTA. Hier addeert men den quadraet van de helfte van 't geral daer het Cossisch teken by staet, tot het ledigh geral: Om dieswille dat het teken + is.

36 Soeckt

36 Soeckt een sodanigh getal/ dat als ick tot het ſ synes quadraets 6 addeere, ofſen soo veel over de 18 komt/ als het gevonden getal is onder de 3 ⅟ Viage, Wat getal is het?

Set 1 ✱ Equatie.
't quadraet 1 ⊠
½ ⊠ + 6 is gelijck 1 ✱ + 24
1 ⊠ = 3 ✱ + 108
1½ 2¼ add.
2½ 110¼ den √ is 9 het gesochte getal.

37 Geeft een getal/ wanneer men syn ½ ende ⅓ met malkanderen multiplicéert/ ende van het product subtraheere 3 + √ 18/ dat het overblyfsel 3y gelijck het gevonden getal.

Set het getal 3y 1 ✱ soo worden ½ ⊠ − 3 − √ 18 ghelijck 1 ✱ ofte 1 ⊠ − 18 − √ 648 = 6 ✱, dat is/

Equatie.

1 ⊠ = 6 ✱ + √ 648 + 18
de ⅟ 3 add. 9
in sich 9 √ 648 + 27 hier uyt den √
de ⅟ √ 162 + 13½
in sich 182¼
trecht 162
rest 20¼ den √ is 4½ addeert ende subtraheert 13½ komen √ 18 + 3 daer by vierdy de √ van 't getal dare het Cossisch teecken by staet dats 3 komt
Facit 6 + √ 18 het gesochte getal.

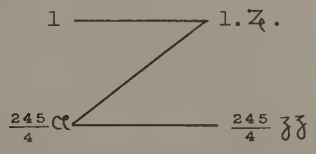
38 Geeft twee ghetalen/ waer van het eene is 4 meer dan het andere/ wanneer men die met malkanderen multiplicéert/ datter komt √ 25 ✱ + 12: Wat getalen zijn datter?

Het

1 ra.	x
$\frac{2}{3}$ ra.	$\frac{2}{3}x$
$\frac{2}{3}$ P. egaux à 150	$\frac{2}{3}x^2 = 150$
75	
225	x = 15
15	

The notation for the second power "P" is peculiar to Forcadel, having been found nowhere else. Schagen used a small p. As used by Scheubel it is a symbol for potestas or for primus multiplication.

In RECORDE'S algebra, The Whetstone of Witte (1557), as has been mentioned previously, the most original and historically important contribution is the sign of equality. His terminology shows the influence of Scheubel. In his treatment of the equation the author has largely drawn from Stifel. The text includes such a variety of interesting problems that the choice of a typical one has been rendered rather difficult. If this discussion were not limited to quadratics, the first choice would have fallen on the problem, "A question of Bricke," as that problem has three distinct points of interest; first, the fact that bricks were sold by the yard; second, the use of the term Triplasesquialtera, and finally, the manner of setting up a relationship which is based on the condition of the problem (p. Hnj^{V}) as



Because this discussion must be limited, the solution of this problem, "whose zensisensike roote is 2," is not given. The problem on "A question of iourneying," (p. Fi. iiij) is selected instead.

Master: What saie you then to this question? There is straunge iorneye appointed to a manne. The first day he must goe $1\frac{1}{2}$ mile, and every day after the firste, he must still augemente his iorney, by $\frac{1}{6}$ of a mile. So that his iorney shall procede by an Arithmeticalle progression. Wherby you might be adsured, that it doeth appertaine to that rule. And accordyng to the canons of that rule, must you worke this question. But for your better instruction, I will helpe you in this woorke.

First aunswere to the question, by the common position: And saie that the tyme of his iorney is .1.Z. of daies. And then shall all the excesses (whiche maie also be called the nomber of the spaces) be .1.Z.....1℄. The common excesse was supposed to be .$\frac{1}{6}$. of a mile. And therefore the somme of all the excesses muste bee $\frac{1Z-1℄}{6}$ that is to saie, the number of all the excesse multiplied by $\frac{1}{6}$, that is here, the first parte of the nomber of the excesses.

And bicause that the firste nomber is .$1\frac{1}{2}$. I must adde it unto the somme of the excesses, and so have I the laste nomber of that progression. Wherefore addyng .$1\frac{1}{2}$. (whiche is $\frac{3}{2}$, or in like denomination with the other, $\frac{9}{6}$) with $\frac{1Z-1℄}{6}$ it will make $\frac{1Z+3℄}{6}$. And that is the laste nomber of the progression.

Now you remember, that in progression Arithmeticall, if you adde the firste nomber to the laste: and multiplie that totalle, by the nomber of halfe the places, there doeth amounte the somme totalle of that progression.

And therefore in this example, if you adde .$1\frac{1}{2}$ (whiche is the firste nomber in the progression) unto $\frac{1Z+8℄}{6}$ (that is the laste nomber of the progression) there wil amounte $\frac{1Z+17℄}{6}$, which beeying multiplied by the nomber of halfe the places, that is $\frac{1}{2}Z$. (For all the nomber of places is .1.Z.) there will rise, $\frac{1ℨ-17Z}{12}$, which is the totalle somme of all the miles: and therefore is equalle to .2955.

Scholar. All the reste, and this againe can I dooe now. Seying $\frac{1ℨ+17Z}{12}$. is equalle to .2955. I will first bryng the whole number to the like denomination, with the fraction, and it will bee $\frac{1ℨ+17Z}{12} = \frac{35460}{12}$. And the omittyng the like denominations .1.ℨ—17.Z. = 35460.℄. That is by translation .1ℨ. = 35460.℄. —17Z. whose roote in value I shall finde out thus. I multiplie $\frac{17}{2}$ squarely, and it will be $\frac{289}{4}$, which is a square nomber, and hath for his roote $\frac{377}{2}$, from which I shall abate $\frac{17}{2}$, and then remaineth $\frac{360}{2}$, that is .180. whiche is the value of .1Z. And expresseth the nomber of dayes, which the question requireth.

The proofe. Master. The proofe is this, and the like questions, is, to set foorther the progression with all his termes. Excepte you will for shortnesse, sette downe the first terme, whiche in this example is .$1\frac{1}{2}$: and then by the nomber of the excesses, or distaunces, (whiche is ever one lesse than the nomber of places) multiplie the quantitie of one excesse: and put to it the first terme: and so have you the laste terme. Then havyng the firste terme and the laste, with the nomber of excesses you knowe how to finde the totalle.

As in this example, the nomber of excesses beeyng 179. And the quantitie of one excesse beyng $\frac{1}{6}$, their multiplication giveth $\frac{179}{6}$, unto whiche if you adde the first nomber, that is $1\frac{1}{2}$, it will be $\frac{188}{6}$. And that is the laste nomber of that progression. Then to trie the totalle somme of the miles, adde the first nomber. $1\frac{1}{2}$ to the laste, and thei will make $\frac{179}{6}$ (not 197 as in the original), that you shall multiply by halfe the number of the places, whiche in our example are .90 (sith the whole nomber is .180) and there will amounte .2955. Accordyng as the question saieth.

Scholar. This is sufficient for this question. And at some idle time, I will not sticke to trie it out, by settyng the progression foorthe at large. In the meane tyme I praie you for better exercise, give me some moare questions.

The remaining problems are cubics, which Recorde calls questions of "straunge equation," and biquadratics whose solutions are given in one roote only. This part of his work is ended by a promise to his Scholar (p.Ll. ij^r):

But of these and many other verie excellente and wonderfulle woorkes of equation, at an other tyme I will instructe you farther, if I see your diligence applied well in this, that I have taught you. And therefore here will I make an eande of Cossike nombers, for this tyme.

Robert Recorde was not a great mathematician but the service he rendered to mathematical science was no small one. The textbooks which he wrote were most excellent treatises from a pedagogical point of view and, what is quite as important, they were written for Englishmen in the mother tongue. That the combination of clear treatment and English language was a popular one is attested by the widespread and long continued use of his works.[171]

MENHER DE KEMPTEN, who adopted the symbolism of Christoff Rudolff, demonstrates the solution of the types of quadratic equations and accompanies each by very clear geometrical demonstrations in Practiqve Pour Brievement, 1565, Tij^r.

A simple illustration of one of the algebraic solutions is as follows:

Si $4\mathcal{Z}+12\mathcal{R}+10$ sont egaux à 50, combien vaut $1\mathcal{R}$? Responce: tirez de chascune partie vn, & resteront $4\mathcal{Z}+12\mathcal{R}+9$, egaux à 49, plus tirez de chascun nombre $\surd$, a scauoir de 49, il	If $x^2 + 12x + 10 = 50$, what is the value of x?
	Response: Take one from each side, leaving
	$4x^2 + 12x + 9 = 49.$

en viendra 7, & de $4\mathcal{Z}+12\mathcal{R}+9$. en ceste maniere cherchez la $\surd$ de $4\mathcal{Z}$, qui fait $2\mathcal{R}$, lesquelz mettez à la place du quotient, & il ne restera rien, & puis doublez les $2\mathcal{R}$, & feront $4\mathcal{R}$, & dictes combien de fois ha on $4\mathcal{R}$ en $12\mathcal{R}$, de sorte, qu'en multipliant le nombre produit en soy, qu'on le puisse tirer du nombre absolu, & fera 3 fois, & ne restera rien, & encore multiplie lesdictz 3 en soy, & fersont 9, lesquelz tirez de 9, & ne restera rien, & le quotient fera $2\mathcal{R}+3$ egaux à 7, cy deuant trouuez, ou $2\mathcal{R}$ egaux à 4, fait $1\mathcal{R}$ egal à 2. ou si on tire de chascun 10, il en resteront $4\mathcal{Z}+12\mathcal{R}$, egaux à 40. ou $1\mathcal{Z}+3\mathcal{R}$ egaux à 10. & par la seconde equation sera $1\mathcal{R}$ egaux à 2. Mais si on faut tenir autre maniere, sinon que à la place du signe plus, on y met le signe - . ...

Then, by factoring both sides,

$$2x + 3 = 7$$

and

$$x = 2.$$

This method in which a certain quantity is added to each member of the equation, thereby making them perfect squares, is one of the first of its kind to be found in this period. Later Stevin and Clavius both employ the same method.

Menher's use of the regular form of reduction is found in the following problem:[172]

Si $1\mathcal{Z}$ est egal à $10\mathcal{R} + 20 + \surd 16$.

$$x^2 = 10x + 20 + 16.$$

During its reduction the $\sqrt{16}$ is retained as such until the author takes the square root, "tirez $\surd$," of $45 + \sqrt{16}$, giving 7 as the value of x.

Since ANTICH ROCHA who employed the symbols of Paciolo, based his work on that of Marco Aurel, there is very little necessity for any further comment. Two problems, which will adequately demonstrate his method,[173] will be given here.

Da me numero que multiplicando su metad consu tersu tercio vegnan. .54. pongo el nu.l. co. multiplico. $\frac{1}{2}$ de cosa con $\frac{1}{3}$ de. co. y soldran $\frac{1}{6}$ de ce. yguales a .54. nu. parte nu. por ce. y soldra .l. ce. que valdra .324. nu. cuya ra. q. es. 18. y tanto vale .l. co. y es el numero	Give me a number which multiplied by a half and a third becomes 54.
	$\frac{1}{2}x \cdot \frac{1}{3}x = 54$
	$\frac{1}{6}x^2 = 54$
	$x^2 = 324$
	$x = 18$

[171] L. C. Karpinski, "The Whetstone of Witte" (1557) in Bibliotheca Mathematica, Vol. III, 1913, p. 228.
[172] Practiqve Pour Brievement, 1565, p. Tiij.
[173] Arithmetica, 1565, p. 264^r.

demandado: porque la metad de .18. son. 9. los quales multiplicados por. 6. (que es el tercio de. 18.) foldran 54.

The second problem[174] illustrates the use of mas., more or plus.

Give me two numbers such that one, which is 4 more than the other, multiplied by the other gives 117.

After setting 1.co. for one of the numbers, he arrives at the quadratic equation,

1.ce.mas.4.co. ygual a·117.

$$x^2 + 4x = 117$$

whose solution yields 9 for the lesser number and 13 for the greater number.

The Libro de algebra (1567) of NUÑEZ bears in the Dedication the date December 1, 1564. "The manuscript was first prepared in Portugese some thirty years previous to the preparation of this Spanish translation by Nunez."[175] The author draws entirely from Italian sources, making mention of Paciolo, Tartaglia and Cardan. This problem is typical (p. 170^{v}).

Divide 10 into two parts such that the sum of their square roots is equal to 4.

He then goes on to say that manifestly the two parts could not each be 5, for the $\sqrt{20}$ is not 4. He thereupon concludes that one part must be greater than 5, assigning 5.$\bar{p}$.1.co., or 5 + x, to represent it, and 5.$\bar{m}$.1.co., or 5 - x, to represent the lesser. Consequently, the equation becomes

4.R V.5.$\bar{p}$.1.co.$\bar{p}$. R V.5.$\bar{m}$ 1.co. yguales a 4

or, $\sqrt{5 + x} + \sqrt{5 - x} = 4$. Nuñez now directs the squaring of both sides, giving 16 for one member and 10.$\bar{p}$.R V.100 $\bar{m}$.4.ce. for the other member, that is

$$16 = 10 + \sqrt{100 - 4x^2}$$

or, $$6 = \sqrt{100 - 4x^2}$$

which, after squaring again, yields

$$36 = 100 - 4x^2$$

arriving at $x = 4$.

He concludes that 5 - 4 or 1 is the value of the lesser part and 5 + 4 or 9, that of the greater part.

Nuñez who was well acquainted with the works of Cardan, Tartaglia, Peletier, and other contemporaries, incorporated into his text what was best in each of the works he had studied. The problem just cited attests to the high value that must be ascribed to his work so worthy of the esteem it enjoyed in the 16th century.

PEREZ DE MOYA entitles the section on algebra as "Trata de la substil, y muy ingeniosa regla de Algebra" or "regla de la cosa." After 139 pages of preliminary matter Moya begins the study of quadratic equations in Chapter 53 of the seventh book. Here he explains the types of equations expressing a relationship between the number, the unknown and its square.

The first problem given bears a striking resemblance to those of the other authors already considered. It reads,[176]

Give me a number which added to its power, or square, makes 42.

1 ce.p.1 co. yg. a 42n. $x^2 + x = 42$.

Accompanied by a lengthy explanation, the solution, abbreviated, is as follows:

$\frac{1}{2}$ of 1 (coefficient of x) is $\frac{1}{2}$

$$(\tfrac{1}{2})^2 = \tfrac{1}{4}$$

$$x^2 + x + \tfrac{1}{4} = 42\tfrac{1}{4}$$

$$\sqrt{42\tfrac{1}{4}} = 6\tfrac{1}{2}$$

$$x = 6\tfrac{1}{2} - \tfrac{1}{2} \text{ or } 6$$

The author does not mention the negative root.

According to SMITH, "the Arithmetica of Perez de Moya is considered one of the

[174] Arithmetica, 1565, p. 266^{v}.
[175] Cajori, Notations, p. 162.
[176] Trattado de Mathematicas, 1573, p. 569.

most noteworthy treatises on mathematics produced in Spain in the 16th century."[177]

BRUCAEUS, a professor of mathematics and medicine, bases his discussion of the three general types of quadratic equations on the ordinary geometrical explanation of Al-Khowarizmi.

A brief but very good elementary exposition of algebra was given by G. GOSSELIN in his De arte magna. Although the + and - of Scheubel must have been known to the Frenchmen, Gosselin follows the Italians in his symbols for plus and minus, but he capitalizes them.

BOSMANS furnishes an interesting problem in which is found a close approach to the modern solution of radical equations.[178]

The problem states that two have certain amounts of money unknown to me. The first said to the second, if you give the square root of your amount I will have 3 more than you. But, said the second if you give me the square root, "quadratum latus," of yours I will have 5 more than you. How much money has each?

Gosselin then assigns 1Q or x^2 for the amount that the second has, and, according to the conditions of the problem, the first has 1QM1LP3 or $x^2 - x + 3$. He finally arrives at the equation LV1QM2LP3 aequale 8M2LMLV1QM2LP3 or

$$\sqrt{x^2 - 2x + 3} = 8 - 2x - \sqrt{x^2 - 2x + 3}.$$

Simplifying the equation first by transposition

8M2L aequalia 2LV1QM2LP3 or

$$8 - 2x = 2\sqrt{x^2 - 2x + 3}$$

and then

$$8 - 2x = \sqrt{4x^2 - 8x + 12}.$$

Gosselin squares both sides,

4QM8LP12 aequalia 64M32LP4Q

or

$$4x^2 - 8x + 12 = 64 - 32x + 4x^2$$

$4x^2 - 8x$ is then subtracted from both sides, leaving

52M32L aequalia M8L

that is, $52 - 32x = -8x$

and again removing $-8x$ from each side the equation becomes

52M24L aequalia nihilo

or, $52 - 24x = 0.$

The value of x is $2\frac{1}{6}$, and $\therefore x^2$ is $4\frac{25}{36}$, the amount the second one had, while the first one had 1QM2LP3 or $3\frac{13}{36}$.

The freedom with which Gosselin handles radical equations bespeaks his great familiarity with the subject of algebra as known in that period. His equating of a member to zero or to negative terms is worthy of special note.

The reader's attention is called further to the symbol for the square root sign, LV; that is, the symbol for the unknown followed by the V, for universal root. Salignacus' use of the lower case l for the unknown is strikingly similar.

PROFESSOR BORTOLOTTI of the University of Bologna has made a very careful study of BOMBELLI'S Algebra in connection with unpublished manuscripts. Consequently, we need but briefly cite here an example or two to illustrate Bombelli's notations and his manner of dealing with types of quadratic equations.

The first one of the type, $ax^2 = bx + N$ is taken from the section entitled, "Capito di potenze eguali a tanti, e numero,"[179] or, "Chapter on squares equal to roots and numbers."

...come se si hauerà da, agguagliare 1 ☉ à 12 ☌ p.11. Piglisi il mezzo delli Tanti, ch'è 6, il suo quadrato è 36 che gionto à 11. fa 47, che pigliato il suo lato è R.q. 47 & aggiontoli il mezzo delli Tanti, fa R.Q. 47 + 6, è questo è la ualuta del Tanto.

$x^2 = 12x + 11$

$\frac{1}{2}$ of 12 = 6

$6^2 = 36$

$36 + 11 = 47$

$x = 6 + \sqrt{47}$

[177] D. E. Smith, History of Mathematics, 1923, Vol. I, p. 345.
[178] Eneström, Bibliotheca Mathematica, Vol. 7, p. 56.
[179] L'Algebra Opera, 1579, p. 257.

Problem LI[180] is as follows:

Divide 20 into two parts such that the excess of one square over the square of the other will be 120.

He then takes 10.p.1$\overset{1}{\smile}$, or 10 + x for one part and 10.m.1$\overset{1}{\smile}$, or 10 - x for the other part. After squaring each part, he equates their difference to 120. The excess of 1$\overset{2}{\smile}$ p.20$\overset{1}{\smile}$ p.100 over 1$\overset{2}{\smile}$ m.20$\overset{1}{\smile}$ p.100 is 40$\overset{1}{\smile}$ or $x^2 + 20x + 100$, over $x^2 - 20x + 100$ is $40x$, which is equal to 120. Therefore x is 20 and the greater part is 13, with 7 for the lesser part.

"Bombelli displays as great ingenuity as his predecessors in the solution of the equation"[181]

1$\overset{1}{\smile}$. m. R.q.8$\overset{1}{\smile}$ Eguale à 6	$x - \sqrt{8x} = 6$
1$\overset{1}{\smile}$. m. R.q.8$\overset{1}{\smile}$ p.2. Eguale à 8	$x - \sqrt{8x} + 2 = 8$
1$\overset{1}{\smile}$. m. R.q.2 Eguale à Rq.8	$x - \sqrt{2} = \sqrt{8}$
1$\overset{1}{\smile}$. Eguale à Rq. 8.p.R.q.2.	$x = \sqrt{8} + \sqrt{2}$.

Bombelli's notation, an extension of Chuquet's, is the first approach to exponential symbolism in this period. Bombelli simply introduced positive integral exponents, without writing the base to which they belonged. He symbolized a radical by the use of capital L before the expression, indicating the last term of the radicand by following by a reversed L,[182] as

$\sqrt{23 + \sqrt{8}}$ is written R.q. ⌊23. p.R.q. 8⌋.

Bombelli recognized negative and imaginary roots.

Many of the problems incorporated in his Algebra are taken directly from Diophantus. Although he does not distinguish them from his own, he adheres so closely to the original that a student of Diophantus will find no difficulty in identifying them.

There can be no question that VISCONTI (1581) was influenced in his algebra by Buteo. On page 129 we find "De tribus compositis canonibus" which are identical with those in Buteo's Logistica. Then, too, in the examples illustrating these rules, Visconti uses Buteo's phrases as ductus, contentum, contenti, tetragonicum latus, etc. For example, on the same page as above, we find the following problem, which is identical to one found in Buteo's Logistica (1559).

Numerum inuenire cuius quadratum, cum suis quatuor lateribus, & additione 7, faciat 52.	Find a number which squared with four times itself added to 7 makes 52.

Pono talem numerum esse 1 co. ipsius quadratum est 1 ce. addo quatuor ipsius latera, que sunt 4 co. fit 1 ce. p. 4 co. addo 7. fit 1 ce. p. 4 co. p 7. que omnia proponuntur facere 52, quod est contentum, facio equationem, auferens vtrinque p 7. restat 1 ce. p. 4 co. continens equale contento 45. Et talis terminatio fit semper in canone primo, scilicet in continente quadratum, cum numero linea, & in contento solum numerus. Huiusmodi quantitates ad inuentionem quaesiti tractantur hoc modo.

x, the number

$x^2 + 4x + 7 = 52$

$x^2 + 4x = 45$

$x^2 + 4x + 4 = 49$

$x + 2 = 7$

$x = 7 - 2$

or 5

Primum ex numeri dimidio linearum factum quadratum additur contenti numero, cuius summae tetragonicum latus sumitur, & ex eo latere sublato linearum dimidio, restabit quesitum. Velut in proposito. Quadro dimidium numeri linearum, scilicet 2. sit 4. addo numero 45, fit 49. huius tetragonicum latus est 7. ausero dimidium numeri linearum, quod est 2. restat 5, numerus quesitus. Haec operandi forma perpetuo seruatur in canone primo.

Si autem iuncto quadrato dimidij linearum ad contenti numerum, sumi non possit ex ea summa tetragonicum latus, vt quae non sit quadratus numerus, tunc impossibile est fieri per numeros quod proponitur vt si mutetur datus numerus 52 in 53, tunc equatione facta habeo 1 ce. p. 4 co. aequalem 46, addo 4 quadratum dimidij linearum, ad 46. fit 50. qui non habet tetragonicum latus, igitur nullus est numerus qualis proponitur. Aliqui tamen per lineas ineffabiles respondere solent.

Dicunt enim latus tetragonicum 50 esse R. 50 ⁄ .2, nam ad propositum de numeris lineam non respondere debemus.

[180] L'Algebra Opera, 1579, p. 450.
[181] Ibid., p. 258.
[182] Ibid., p. 252.

The author also uses the idea of an "impossible" solution for irrational roots.

THOMAS MASTERSON (1592) who introduces his text with

> Nothing without labour.
> All things with reason.

divides it into three parts, or "books," the third book of which is devoted to algebra. This is chiefly devoted to the fundamental operations. There are no equations in the text. However, the methods for extracting roots are similar to those employed for solution of equations.[183]

If this ȝ number 321489 be given to have his ȝ roote (square root) extracted.

Masterson then uses the "pricke" method of extraction similar to that found later in Praxis analytica by Harriot. And again,

If 18 + √ȝ 128 (18 + √128) be the ȝ of a binome, and his ȝ roote is required, the ½ of the last particular is √ȝ 32, his ȝ is 32, and 18 cannot be parted or divided into any two numbers which multiplied together produce 32, but only in 16 and 2, therefore √ȝ 16 + √ȝ 2, that is 4 + √ȝ 2, is the roote demanded.

The Stratioticos

Long since attempted by Leonard Digges Gentleman, Augmented, digested, and lately finished, by Thomas Digges, his Sonne.

shows some influence of Stifel. Although he applied his algebra to military problems, his method is interesting in its comparative brevity. The following is an example of the fifth rule:[184]

1 ℳ ℋ 14 ℳ - 33	$x^2 = 14x - 33$
The Moytie of the number of Primes is 7,	½ of 14 = 7
that squared maketh 49,	$7^2 = 49$
from this I deduct 33,	49 - 33 = 16
the abstract number, resteth 16,	$\sqrt{16} = 4$
whose Root 4 added to 7, the Moytie Fundamentall, maketh 11, the greater Roote, deduct the same 4 from 7, resteth 3 the lesser Radix. The truth whereof is thus apparant,	$x = 7 + 4$ or 11, the greater root $x = 7 - 4 = 3$, the lesser root

square ariseth 121, the Square which should be equall to 14 Rootes less 33, 14 times 11 maketh 154 the number of the Rootes, from this deduct 33, the abstract number resteth 121 your square. In like sort, y lesser Roote 3 squared, maketh 9. Now 14 of these Rootes are 42, from which that both the one and the other are true Rootes of this AEquation, and more than these it is impossible to finde.

Here again both roots, if positive, are considered.

There is a resemblance between the notations of Cataldi, 1610, and those found in the Stratioticos in that the authors used the same scheme for representing the powers of the unknown.

The Dutch books commonly present new and interesting material. PETRI presents a problem that gives a digression from the "Moytie" type of solution, as follows, with one of the early instances of factoring. (fol. 146, recto 147 verso, ed. 1605.)

Item soo 9z + 24ℓ + 20 gelijck zijn 173 hoe vele is 1ℓ. antwoort trecket van elcker partie 4 sal noch resten 9z + 24ℓ + 16 gelijck 169/ trecket nu van elcker partie √ te weten van 169 sal comen 13 en van 9z + 24ℓ + 16 in deser manieren soecket den √ wt 9z welcke is 3ℓ die settet int quotient daer na so dupliceert de 3ℓ sal comen 6ℓ. en spreket hoe menichmal 6 ℓ in 24ℓ. sulcker wyse dat somen het getal product in hen seluen quadrate multipliceert dat men het nemen mach vant getal absolut en sal wesen 4 mael en sal het quotient van dese extractie maken 3ℓ + 4 gelijc 13 en 1ℓ sal dan zijn 3	$9x^2 + 24x + 20 = 173$. Take 4 from each side, giving $9x + 24x + 16 = 169$. Taking the square root of each side, gives 13 for the right member. The square root (√) of $9x^2$ is 3x which, duplicated, gives 6x·24 + 6 is 4. Therefore 3x + 4 = 13 and x is 3.

9ȝ + 24ℓ + 16 | 3ℓ + 4

8

4

Ofte so men trecket van elcker partie 20 restet noch 9ȝ + 24ℓ gelijck	However many times one takes 20 from each part there shall

[183] Arithmeticke, 1592.

[184] Stratioticos, 1594, p. 51.

153 ofte $1\mathcal{Z} + 2\frac{2}{3}\mathcal{Q}$ gelijck 17 ende deus die tweede aequatie sal $1\mathcal{Q}$ maken 3 als voren.

Maer so men wilde treckon $\sqrt{\mathcal{Z}}$ van $9\mathcal{Z} - 24\mathcal{Q} + 16$ / so en doet anders met dan vorens geleert dan inde plaetse van het teken + men alhier setten moet het teken - .

remain $9x^2 + 24x = 15$ or $1x^2 + 2\frac{2}{3}x = 17$ and thus the second equation makes $x = 3$.

But if one should wish the square root of $9x^2 - 24x + 16$, then he does not do it differently than he learned previously. Then in the place of the sign + , he must put the sign - .

In the last part he explains that 20 could be taken from both sides, leaving $9x^2 + 20x = 153$, and that after dividing through by 9 the equation becomes $x^2 + 2\frac{2}{3}x = 17$ and $x = 3$ as before. He also adds that, if the middle term is minus, one must take the square root as before but instead of the + use -.

It is noteworthy that Petri makes use of zero coefficients, particularly in regard to cubics which he wished to factor. For example in factoring $x^3 - 64$, he wrote this binomial as $x^3 + 0x^2 + 0x - 64$.

Another problem has $x + 3 = \sqrt{14 + \sqrt{180}}$, whose left member is expressed as "$\sqrt{\mathcal{Z}}$ van $14 + \sqrt{180}$." He squares each side, giving $x^2 + 6x + 9 = 14 + \sqrt{180}$ or $x^2 + 6x = 5 + \sqrt{180}$. Then the square of $\frac{1}{2}$ of 6, added to $5 + \sqrt{180}$, results in $14 + \sqrt{180}$.

In Fol. 149 verso, ed. 1605,

Daer van $\sqrt{\mathcal{Z}}$ sal comen $3 - \sqrt{5}$ daer van trecket die helfte van die middelste quantiteyt welche 3 ende sal noch resten $\sqrt{5}$ voor de weerde van $1\mathcal{Q}$.

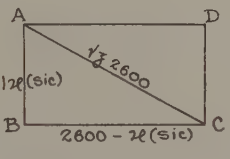
Of this the square root should be $3 + \sqrt{5}$. From this take the half of the middle term, which is 3, and there shall remain $\sqrt{5}$ for the value of the root.

Examination by PROFESSOR KARPINSKI of Menher and Petri shows that Petri copied directly from Menher with change only in the illustrations. The earlier of the English texts, namely, The Pathway of Knowledge by W. P(hilips) follows Petri; and Tap, in the text of the same title, follows Menher, as he himself states. The similarity between these two editions of The Pathway of Knowledge is due to the fact that Petri copied Menher.

The Arithmetica Philosophica (1608) by PETER ROTH treats mainly of cubic equations, a result perhaps of his great interest in the work of Cardan. A few solutions of quadratics, however, are found in the text; particularly in geometrical applications.

CLAVIUS was a German who spent much time in Rome. His algebra marks the appearance in Italy of the German + and - and of the algebraic signs used by Stifel. He refers to the notations 1q, 2q, etc., to represent unknowns as having been used by Cardan, Nonius, and others. His own notation and method are well illustrated by a geometrical application given in Chapter XXXII. Although it is not his simplest problem, it, nevertheless, gives one an idea of the use Clavius made of two roots and negative coefficients.[185]

In a rectangle ABCD, the area contains 120 "palmos" and the diameter, or diagonal is $\sqrt{\mathcal{Z}}$ 2600 ($\sqrt{2600}$) "palmos" long. Find the lengths of the sides.



Since the sides AB, BC, squared, are equal to the diameter AC squared, which is 2600, and AB squared is $1\mathcal{Q}$ (x), BC squared will be 2600 - $1\mathcal{Q}$(2600 - x). Therefore, the sides themselves are $\sqrt{\mathcal{Z}}\, 1\mathcal{Q}(\sqrt{x})$ & $\sqrt{\mathcal{Z}}(2600 - 1\mathcal{Q})$, ($\sqrt{2600 - x}$) whose product is the area BD, namely, 120 "palm." In geometrical practice, the sides squared, $1\mathcal{Q}$ and $2600 - 1\mathcal{Q}$, multiplied together give 14400, which is the square of the area. When $1\mathcal{Q}$ is multiplied into $2600 - 1\mathcal{Q}$, $2600\mathcal{Q} - 1\mathcal{Z}$, ($2600x - x^2$) will be equal to 14400. Adding $1\mathcal{Z}$ to each side results in $1\mathcal{Z} + 14400$ equal to $2600\mathcal{Q}$, ($x^2 + 14400 = 2600x$). Subtracting 14400 from each side, $1\mathcal{Z}$ will be equal to $2600\mathcal{Q} - 14400$. Now $\frac{1}{2}$ of 2600 is 1300 which squared is 1690000 and subtracting 14400, leaves 1675600. If to the square root of this number $\sqrt{\mathcal{Z}}$ 1675600 we add the half 1300, $1\mathcal{Q}$ will be $1300 + \sqrt{\mathcal{Z}}$ 1675600 ($1300 + \sqrt{1675600}$). And if this same root is subtracted from the 1300 (since this equation has a double root, on account of the minus sign by which the absolute number is

[185] Algebra, 1609, p. 357.

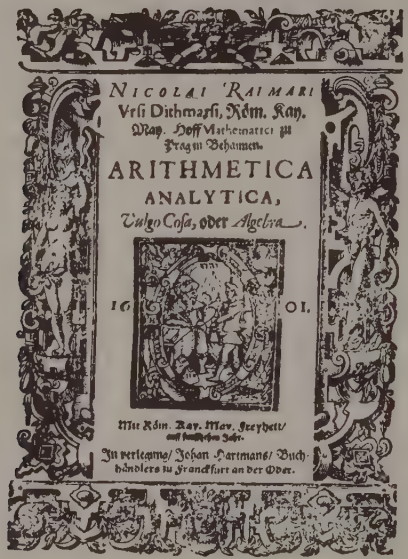


Fig. 37. Title page of Nicolaus Reimarus Ursus, Arithmetica Analytica, Vulgo Cosa, oder Algebra (Frankfurt, 1601).

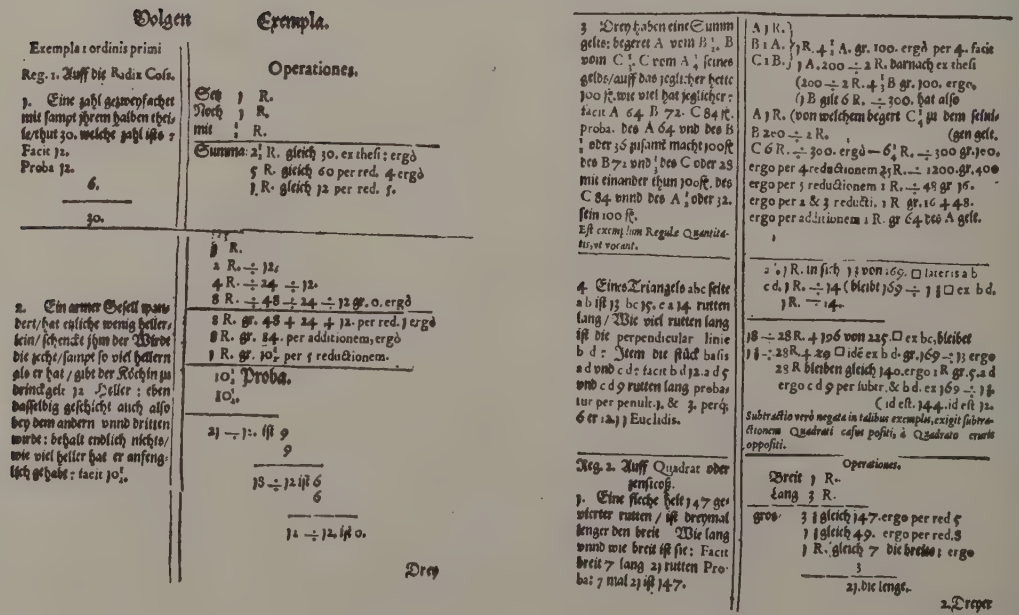


Fig. 38. Two pages from the above text demonstrating his solution of simple linear and quadratic equations.

characterized), the lesser side AB squared will be 1300 - $\sqrt{z}$ 1675600 (1300 - $\sqrt{1675600}$), but the greater side BC squared will be 1300 + $\sqrt{z}$ 1675600, which is also the remainder if the square of the side AB, 1300 - $\sqrt{z}$ 1675600, is subtracted from 2600, the square of the diameter AC, as appears in this formula:

2600 + $\sqrt{z}$ 0	One, therefore, sees
1300 - $\sqrt{z}$ 1675600	two squares 1300
1300 + $\sqrt{z}$ 1675600	- $\sqrt{z}$ 1675600, and 1300 + $\sqrt{z}$ 1675600, as making

2600, the square of the diameter. And if the sides, of each $\sqrt{z}$ (1300 - $\sqrt{z}$ 1675600) and $\sqrt{z}$(1300 + $\sqrt{z}$ 1675600), ($\sqrt{1300 - \sqrt{1675600}}$ and $\sqrt{1300 + \sqrt{1675600}}$), are multiplied together, the result will be the area of the rectangle, or $\sqrt{z}$ 14400, which is "120.palm."

Clavius then gives an alternative method by assigning 1 $\mathcal{x}$ to the shorter side AB and 2600 - 1 $\mathcal{x}$ to the greater side. Or, again, he lets AB be 1 $\mathcal{x}$, and BC, 2600 - 1 z, and completes the solution under this new assignment.

Clavius and Petri were the first in this period to employ zero coefficients, as in the illustration given above and again in problem 31 of Chapter XXX, 1609, when Clavius wishes to subtract:[186]

$$\begin{array}{r} +0 \;+\; 0\mathcal{x} + 1 \\ 900 - 60 + 1 \\ \hline -900 + 60 + 0z \end{array}$$

The Tratado de la Algebra, 1619, by TOLRA, devotes Chapter IX to the simple linear equation, thereby giving thirty-four pages to applications of this type of equation. Chapter X takes up the pure quadratic; the next two chapters deal with the pure cubic and biquadratic equations, respectively. It is the last four chapters of the text that discuss the affected equations, with Chapter XIII devoted to the type $ax^2 + bx = N$; Chapter XIV to $ax^2 + N = bx$; the next chapter to $ax^2 = bx + N$; and finally, the last chapter to the affected cubic and biquadratic equations.

The following example (p. 66) will adequately illustrate the method used by Tolra in solving all types of affected quadratic equations.

Dadme dos numeros, que el uno sea 4 mas $\bar{q}$ el otro, y multiplicando el uno con el otro vengan 117. Pon, $\bar{q}$ el menor sea 1 $\mathcal{x}$, el mayor sera 1 $\mathcal{x}$ + 4 $\mathcal{Q}$: estos multiplicalos por 1 $\mathcal{x}$, que es el numero menor, y vendra 1 z + 4 $\mathcal{x}$, igual a 117 $\mathcal{Q}$. Sigue la regla: parte $\mathcal{x}$ por z, y $\mathcal{Q}$ por z, esto es 4 por 1, y 117 por 1, y vendran dos quocientes, el del Caracter mediano sera 4, y el del menor 117. La metad de 4 (que es el quociente del mediano) son 2, multiplicados en si son 4, juntados con 117 (que es el quociente del menor) son 121. Cuy a rayz quadrada es 11, de los quales quita 2 (que es la metad del quociente del mediano), y quedaran 9. Tanto vale la $\mathcal{x}$, y numero menor, el mayor sera 13: y multiplicados 13 por 9, vienen 117, como dize la pregunta.

Give me two numbers such that one will be 4 more than the other and their product will be 117. Let the smaller number be x, and the greater x + 4. Their product is $x^2 + 4x$ which is equal to 117. According to the rule, divide x and the constant by 1 (the coefficient of x^2). Then $\frac{1}{2}$ of 4 (which is the coefficient of the middle term) is 2 which squared is 4, and added to 117 makes 121. The square root of 121 is 11. From this 2 is subtracted, leaving 9. This is the value of the smaller number; the larger will be 13 and 13 times 9 will give 117 as was required.

Negative roots are not considered by Tolra. Since the theory of equations is applied immediately to practical problems, one might hesitate to make the statement, had not Tolra himself said, in discussing one type of equation, (p. 69) that, in the case on hand, it is impossible to subtract 5 from 3.

Tolra's frequent allusion to Marco Aurel justifies the conclusion that the latter exerted no small influence on the author of the Tratato de la Algebra.

FOLLINUS has freely embodied the

[186]This present finding necessitates the correction of a statement in Miss Vera Sanford's Short History of Mathematics, p. 168, to the effect:

> In view of the present emphasis given to the solution of quadratic equations by factoring, it is interesting to note that this method was not used until Harriot's work of 1631.

This is erroneous, since it has just been shown that this method was employed as early as 1608 by both Petri and Clavius.

ideas of Clavius in his Algebra, even incorporating problems and their solutions as given by his predecessor. But one problem (p. 234) is given here in order to review his method and style and to call attention to the use of the word planus.

Tempus temporis triplum est, minorisque $\frac{1}{4}$ per $\frac{1}{5}$. maioris facit 30. quodnam est vtrumque tempus?

Esto 1φ. pro primo: Ergo pro secundo 3φ. quia hic numerus triplus est primi. Est autem $\frac{1}{4}$. primi $\frac{1}{4}\varphi$. & $\frac{1}{5}$. secundi siue maioris $\frac{3}{5}\varphi$. Nunc multiplicentur hi duo numeri inter se, puta $\frac{1}{4}\varphi$ & $\frac{3}{5}\varphi$ planus $\frac{3}{20}$ est aequalis 30. AEquatio reducatur per Multiplicationem in crucem sic $\frac{3\mathcal{Z}}{20} \times \frac{30}{1}$ vnde aequalitas erit inter 3$\mathcal{Z}$ & 600. Diuisis itaque 600. per 3. quoti radix quadrata ostendit primum numerum: quotus est 200. huius radix quadrata est $\sqrt{\mathcal{Z}}$200. pro 1φ. Ergo pro 3φ. inuentus numerus triplicandus est, id est, multiplicandus per 9. vt numeri sint euisdem generis: quare secundus & maior numerus est, $\sqrt{\mathcal{Z}}$ 1800. Quod autem ita res se habeat, facilè probatur. Minoris enim $\frac{1}{4}$. est $\sqrt{\mathcal{Z}}$ $\frac{200}{16}$. & maioris $\frac{1}{5}$ est $\sqrt{\mathcal{Z}}$ $\frac{1800}{25}$. qui numeri inter se multiplicati faciunt $\sqrt{\mathcal{Z}}$ $\frac{360000}{400}$. Diuisione facta numeratoris per nomen, quotus erit $\sqrt{\mathcal{Z}}$ 900. cuius quadrati radix est 30. nam tricies triginta faciunt 900. Quare operatio Algebraica rectè hic se habuit.

x = the first time

$3x$ = the second time

$\frac{1}{4}x \cdot \frac{3}{5}x = \frac{3}{20}x^2$

$\frac{3x^2}{20} = 30$

$\frac{3x}{20} \times \frac{30}{1}$

$3x^2 = 600$

$x = \sqrt{200}$, or the first time

$3x = \sqrt{1800}$, or the second time.

On page 66 of his Algebra Applicata, (1622), CATALDI, a professor of mathematics and astronomy in Bologna, gives an interesting percentage problem in which the merchant invests 100, accumulating 144 at the end of two years. It will be observed that Cataldi makes no mention of units, simply speaks of them as 100 and 144.

Vn Mercante trafficando 100. si troua in fine di dui anni hauere accumulato 144. si domanda à quanto per 100. l'anno a capo d'anno viene ad hauer guadagnato.

After stating the conditions of this problem, in which he desired to know at what per cent he had to trade 100 in order to accumulate 144 of them in two years, he goes into great detail, somewhat after the style of Feliciano. By letting cosa (x) be the per cent gained on the investment each year, he gives the following solution:

100 douenta 100. piu 1. cosa cioe 1. douenta 1. piu $\frac{1}{100}$ cosa egli douentarà 100. piu 1. cosa egli douentarà 100. piu 2. cose piu $\frac{1}{100}$. censi. Et è eguali à 144. però 2. cose piu $\frac{1}{100}$. censi eguale à 44.

Cioe 200. cose piu 1. censo eguale à 4400.

100
100
10000
gionto 4400
fa 14400 la radite e 120. cauato 100. mita del numero delle cosa resta 20. che e il valore della cosa.

x = rate of investment

$100 + x$, or amount he had at end of first year.

$100 + 2x + \frac{1}{100}x^2$, amount he had at end of second year.

$100 + 2x + \frac{x^2}{100} = 144$

$x^2 + 200x = 4400$

$x = 20$, or rate of investment.

In both editions, the unknown is represented by crossed numbers, illustrated in the discussion of symbols. On page 380 of his Notations, Cajori says that Cataldi "deplored the existence of many different notations for the unknown numbers and their powers, and the inconveniences resulting from such diversity." He points out the difficulty of finding in the ordinary printing establishment the proper type for the representation of the different powers. He proposes to remove both inconveniences by the use of numerals indicating the powers of the unknown and distinguishing them from ordinary numbers by crossing them out, so that $\not{0}$, $\not{1}$, or $\not{\mathcal{Z}}$, $\not{2}$, $\not{3}$, would stand for x^0, x^1, x^2, x^3 ... Such crossed numerals, he argued, were convenient and would be found in printing offices since they are used in arithmetics giving the scratch method of dividing, called by the Italians the "a Galea" method. The reader will recall that Cataldi's notation resembles somewhat that of Leonard and Thomas Digges of England.

Most of the work on algebra taken

up by ALBERT GIRARD in his Invention Nouvelle en L'Algebre (1629) deals with the cubic equation. However, on page 34 is found

Soit 1(2) esgale à 6(1)+40. Notez qu'és operations suivantes il faudra poser des nombres apariez, ainsi que leur prodiuct soit le (0) (comme icy 40) qui fait que lors que la valeur de 1(1) est nombre entier, qu'alors l'operation est souvent tres brieve & facile plus qu'en nulle sorte de reigle.

$x^2 = 6x + 40$

Dividing each term of the equation by x gives

$$x = 6 + \frac{40}{x}$$

2	20
4	10
5	8

factors of 40

Divisions tout par 1(1) viendra 1(1) esgale à $6 + \frac{40}{1(1)}$: C'est a dire que 6 + une des parties aliquotes de 40; (si la solution est un nombre entier) sera la valeur de 1(1).

2	20	Ayant fait une table comme icy a
4	10	
5	8	

efficients de 40

costé, adjoustez le 6 au 2, fait 8, ce qui n'accorde avec 20: Adjoustez le 6 avec le 4, fait 10, ce qui accorde avec le 10; donc 10 est la valeur de 1(1).

If one chooses 2 for x, 6 + 2 is 8, which is not in accord with the other factor 20. If one chooses 4, 6 + 4 is 10 which corresponds to the factor 10. Therefore x = 10.

Only the positive root is considered. Girard uses Ramus' plan by which a (1), (2), ... placed before a number indicates the root and after the number signifies the power.

Farther on in the text he quotes problems from Stevin; he takes others from Francois Vieta's De Recognitione Equationum.

We see in Girard another attempt to extend the notations of Chuquet, Bombelli, and Stevin; the notations of Bombelli and Stevin are only variants of that of Chuquet, who carried the use of the exponential notation to a completeness which received due appreciation in this period. Chuquet does not use fractional exponents, limiting his notation to those involving integral, zero, and negative exponents. Girard, however, adopts the practice of Stevin in using fractional exponents. Thus "$(\frac{2}{3})$ 49" means $49x^{\frac{2}{3}}$. Girard's use of the parentheses, rather than Stevin's circles, may have been necessary to facilitate printing.

JOANNES LANTZ devotes most of Book II of his Institutionum Arithmeticarum, 1630, to the simple equation. The last three problems of this book are solvable as quadratic equations. The last problem (p. 136) deals with two societies, one of which has 16 members more than the other. Each society has 688 coins more than the total number of men in the combined societies to be distributed among its members. In other words, if 1℘ (or x) is the number of members in the smaller society, the membership in the other is 1℘ + 16 (x + 16) and the number of coins distributed in each society is 2x + 704. It is required to find the number of members in each society and amount of money received by each member in the respective society. Those in the smaller society receive 8 more coins than those in the larger group. Lantz arrives at the equation,

$$\frac{22℘ + 704}{1℘}$$

equal to $\frac{10℘ + 832}{1℘ + 16}$ $\left(\frac{2x + 704}{x} = \frac{10x + 832}{x + 16}\right)$

which simplified becomes 10q + 832℘ equal to 2q + 736℘ + 11264, ($10x^2 + 832x = 2x^2 + 736x + 11264$) making 32 the membership of the smaller society and 48 that of the larger; each man in the former society received 24 coins, each in the latter received 16 coins.

HENRION in Sommaire de l'algebre, 1631, passes without any further explanation and demonstration from the fundamental operations directly to problems whose solutions are algebraic. The influence of Clavius can easily be detected even if one were not informed in the chapter on second roots, that he was following Stifel, Peletier, and Clavius. This accounts, no doubt, for Henrion's "conservatism in regard to his literal coefficients in equations."[187]

[187] Cajori, Notations, p. 187.

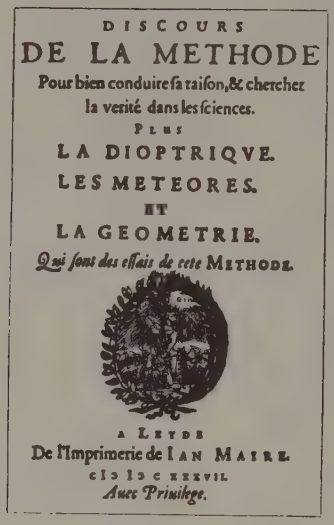

DISCOURS
DE LA METHODE
Pour bien conduire ſa raiſon, & chercher
la verité dans les ſciences.
PLUS
LA DIOPTRIQVE.
LES METEORES.
ET
LA GEOMETRIE.
Qui ſont des eſſais de cette METHODE.
A LEYDE
De l'Imprimerie de IAN MAIRE.
cIↃ IↃ C XXXVII.
Avec Privilege.

Fig. 39. Title page of Descartes, Discours de la Methode (Leyden, 1637).

DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque absolutiſſimis Commentariis illuſtrati.
AVCTORE CLAVDIO GASPARE BACHETO
MEZIRIACO SEBVSIANO, V.C.

LVTETIAE PARISIORVM,
Sumptibus HIERONYMI DROVART, via Iacobæa,
ſub Scuto Solari.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

Fig. 40. Title page of Bachet, Diophanti Alexandrini Arithmeticorum (Paris, 1621).

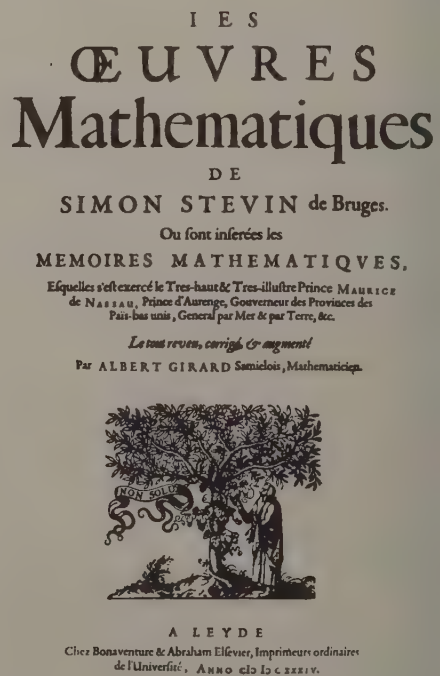

I E S
ŒUVRES
Mathematiques
DE
SIMON STEVIN de Bruges.
Ou ſont inſerées les
MEMOIRES MATHEMATIQVES,
Eſquelles s'eſt exercé le Tres-haut & Tres-illuſtre Prince MAURICE de NASSAU, Prince d'Aurenge, Gouverneur des Provinces des Pais-bas unis, General par Mer & par Terre, &c.
Le tout reveu, corrigé & augmenté
Par ALBERT GIRARD Samielois, Mathematicien.
NON SOLI
A LEYDE
Chez Bonaventure & Abraham Elſevier, Imprimeurs ordinaires de l'Univerſité. ANNO cIↃ IↃ C XXXIV.

Fig. 41. Title page of Stevin, Les OEuvres Mathematiques (Leyden, 1634).

Problem 17 (p. 351) applies algebra to geometry.

Il y a vn quarré duquel le diametre, & le costé ensemble sont 6 trouuer sceux costé & diametre.

Soit posé le costé 1ℝ & partant le diametre 6 - 1ℝ. Et pource que le quarré du diametre est double du quarré du costé: & le quarré du costé est 1 q. & le quarré du diametre est 36 - 12ℝ + 1q, il y aura equation entre 2q & 36 - 12ℝ + 1q. & ostant 1q de chacun demeurera equation entre 1q. & 36 - 12ℝ.

x, the side of the square
6 - x, the diameter.
x^2, side squared
$(6 - x)^2$, diameter squared,
or $(36 - 12x + x^2)$
$2x^2 = 36 - 12x + x^2$
$x^2 = 36 - 12x$

The solution is regular, giving 1R as $\sqrt{72} - 6$, or the side of the square, and $12 - \sqrt{72}$ for the diameter.

BOSMANS answers the question concerning the number of sources which influenced Stevin by observing that Stevin knew the foundations of Greek mathematics, notably those of Euclid and Diophantus. He also knew Al-Khowarizmi, whose algebra circulated in manuscripts, perhaps in the Latin version of Gerard of Cremona, and certainly in manuscripts of Robert of Chester. Stevin himself cites a few of the authors studied, the Portuguese Pedro Nuñez, and the three Italians, Cardan, Tartaglia, and Bombelli. Yet it is strange that, although he had a great admiration for the Germans, he seems to have ignored all the German algebraists, especially Stifel.

STEVIN made a great step in advance of Bombelli by the introduction of fractional exponents, although he does not actually use them. In Stevin's notation a circle is substituted for Bombelli's arc. This is not strictly adhered to, for in his OEuvres of 1634, parentheses or (4), representing x^4, takes the place of the circle. If more than one unknown occurs, Stevin marks the first unknown "1 ⊙," the second "1 secund ⊙," and so on. Another characteristic of Stevin occurs in question XX, page 98, of L'arithmetique, 1634, which reads,

Trouvons un ⓪ tel, que son quarré - 12 multiplié par la somme du double d'icelui (sic) ⓪, & le quarré de - 2 & 4, le produict soit egal au quarré du produict - 2 par icelui ⓪ requis.

Here Stevin requires that a certain number be found which he designates by his symbol for a constant, ⓪, and in the solution he assigns ① to the number.

In the explanation of the general types of equation we find a certain clarity that makes his solution very intelligible. For example in[188]

Premiere Difference De Second Terme ① + ⓪. $(ax^2 = bx + N)$.

Explication du donné. Soient donnez trois termes selon le probleme tels: le premier 1②, le second 4① + 12, le troisiesme 1①. Explication du requis. Il faut trouver leur quatriesme terme proportionel. $x^2 = 4x + 12$

Construction

La moitie de 4 (des 4①) est	2	½ of 4 is 2
Son quarré	4	$2^2 = 4$
Au mesme aiousté le ⓪ donné, qui est	12	12 + 4 = 16
Donne somme	16	
Sa racine quarrée	4	16 = 4
A la mesme aiousté 2 premier en l'ordre faict	6	x = 4 + 2 or 6.

Je di que 6 est le quatriesme terme proportionel requis.

Demonstration Arithmetique. Puis que nous disons 1① valoir 6, doncques par la 66 probleme, 1② vaudra 36, & 4① + 12 vaudront aussi 36; Parquoi mettons soubs chascun terme sa valeur en ceste sorte:

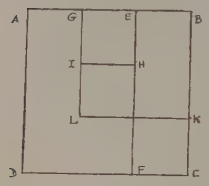
1 ②.	4① + 12.	1 ①.	6
36.	36.	6.	6.

Et appert que 6 est leur quatriesme terme proportionel.

Autre demonstration geometrique.

[188] Les OEuvres Mathematiques, 1634, p. 66.

Soit descript le quarré ABCD, denotant 1②; doncques son costé BC (lequel prouverous valoir 6, à la fin de la demonstration) sera 1①, cai multipliant 1① en soi, faict 1②. Puis soit menee la ligne EF, parallele avec AD, & soit AE 4; Ergo le rectangle AF veuque AD est 1① sera 4①. Or puis que tout le quarré ABCD, qui est 1②, est egal à 4① + 12, & que le rectangle AF faict 4①, le rectangle EC sera 12. Doncques les trois termes donnez en nombres, nous les avons ici en gradeurs, à scavoir ABCD 1②, egale a AF 4① + EC 12: Et BC est le 1①. Or faisons maintenant la construction par ces grandeurs, sembable à la precedente des nombres en ceste sorte: La



moitie de AE 4, qui est GE, est	2
Son quarré GEHI	4
Au mesme aiosté le ⓞ donné, c'est à dire EC	12
Donne somme pour le gnomon HIGBCF, ou pour le quarré GBKL (qui est egal audict gnomon)	16
Sa racine BK	4
A la mesme aiousté GE 2 premier en l'ordre, ou en son lieu KC (car KC est egal à GE) faict pour BC	6

Ce qu'il falloit demonstrer.

This is but a continuance of the geometrical demonstration of Al-Khowarizmi.

Applications of algebraic principles are handled in the same manner as above, with the addition of a column of numbers to the right of the solution which are numerical substitutions fulfilling the conditions of the problems, thereby giving a check to the solution[189]

Trouvous deus nombres tels, que leurs quarrez facent 7, & que l'un quarré subtraict de l'autre, reste 1.	Find two numbers, such that their squares make 7 and the difference of the squares is 1.

Constrvction

Soit le premier nombre requis	1 ①	2
Son quarré pour premier quarré	1 ②	4
Ergo le second quarré (puis qu'avec le premier quarré il doit faire 7) sera necessairement	-1 ② + 7	3
Sa racine quarrée, pour le second nombre requis.	√bino. -1 ② 7	√3
Difference des quarrez	2 ② -7	1
Egale à		

Lesquels termes reduicts, 2② seront egales à 8; Et par le 78 probleme 1① vaudra 2.

Je di, que 2 & √3 sont les deux nombres requis.

Demonstration. Les deux quarrez de 2 & de √3, qui sont 4 & 3, sont ensemble 7. Item soubstraict 3 de 4 reste 1, selon le requis; ce qu'il falloit demonstrer.

Nota. L'ou pourroit encore faire ceste construction sinsi:

Soit le premier nombre requis	1 ①	√3
Son quarré pour le premier quarré	1 ②	3
Ergo le second quarré (puis que du premier il doibt differer en 1) sera +1 or -1 soit	1② + 1	4
Sa racine quarrée pour le second nombre requis	√bino. 1② + 1	2
Somme des quarrez	2② + 1	7
Egale à	7	

Lesquels termes reduicts 2 ② seront egales a 6; Et par le 78 probleme 1 ① vaudra 3. & les deux nombres requis seront comme dessus 2 & √3.	$2x^2 = 6$ $x = 3$ $\therefore$ the numbers are 2 and 3.

HUME, British by birth, is the last of the group of men who by various notations contributed to the development of the modern notation for exponents. Most of Hume's mathematical works were in French, including Algebre de Viete, 1636, nine were in Latin. The addition of exponents is evident in the following:[190]

Soit vne equation entre 8ij - 64 & 16j, a cause que 64 a le signe_, il faut adiouster 64 a tous les deux & vous aurez 8ij, & 16j + 64 egaux: Ainsi 2ij - 3j & 104 seront reduits à 2ij & 104 + 3j.	$8x^2 - 64 = 16x$ or, $8x^2 = 16x + 64$ Also, $2x^2 - 3x = 104$ and $2x^2 = 104 + 3x$

In this discussion these and subsequent equations, noting that ij represents x^2 and j designates x, can be reduced to the form $ax^2 = bx + N$, and solved in the regular manner.

MARTYN WILKEN in his Officina Algebrae, 1636, refers to John Faulhaber,

[189] Les OEuvres Mathematiques, 1634, p. 95.
[190] Traicte de L'Algebre, 1635, p. 34.

Joannes Remmelinus and Peter Roth. His treatment of equations indicates a greater dependence on Faulhaber than on the other authors mentioned. Wilken introduces the study of equations by a series of tables similar to the one in Figure 42. The problems are then stated as follows (p. F3):

Eenen Columnaer-ghetal uyt Trigonalibus, den Agregato Quarto toeslaende/is ghelyck Numero Pyramidali uyt Tetracosi-octaconta-heptagonalien, zynde den derden Corperlycken Ghestachts ghetalen toebehoorende/ende die Radices zyn malkamderen mede ghelijck. Draghe naer den Columnari Numeri? Facit:

11413869531 $\frac{391}{729}$ wiens Radix $76\frac{2}{3}$.

The underlined expressions determine the members of the equation by referring to the proper tables, as, for instance, the left-hand member may be found in Figure 42, corresponding to the fraction which belongs to "Trigonalien." In the same manner the other member of the following equation may be determined:

3 B ʃʒ + 49. ℒ cub + 315. A ʃʒ + 1015. ℒ ʒ + 1722. cub + 1456. ʒ + 480. R. (door 5040) Gelyck
97. A ʃʒ + 486. ʒʒ + 491. cub ÷÷ 474 ʒ ÷÷ 576. R. (door 24)
3. B ʃʒ + 49. ʒ cub + 315 A ʃʒ + 1015. ʒʒ + 1722. cub + 1456. ʒ + 480. R Ghelijck:
20370. A ʃʒ + 102060. ℒ ʒ + 103110. cub ÷÷ 99540. ℒ ÷÷ 120960. R. Subtraheert van elcker sijde 210. ʒʒ + 1260. cub + 2310. ʒ + 1260 R. komt:
3 B ʃʒ + 49. ʒ cub + 315. A sʒ + 805. ʒʒ + 462. cub ÷÷ 854. ʒ ÷÷ 780. R. Ghelijck
20370. A sʒ + 101850. ʒʒ + 101850. cub ÷÷ 101850. ℒ ÷÷ 122220. R.
Nu divideert yeder partye/elck bysonder/door

1. A sʒ + 5. ʒʒ + 5. cub ÷÷ 5. ÷÷ 6. R.

komt

3. ℒ + 34. R + 130. ghelijck 20370
3. ℒ + 34. R. ghelijck 20240.
3. ℒ ghelijck 20240. ÷÷ 34. R.

17.
3
60720. + 289.
289.
√61009. | 247.
. . . | ÷÷17.
230 | $76\frac{2}{3}$

door 3

Facit, $76\frac{2}{3}$ voor die waerde van 1.R.

This solution is interpreted in the following manner:

$$\frac{3x^7 + 49x^6 + 315x^5 + 1015x^4 + 1722x^3 + 1456x^2 + 480x}{5040}$$

$$= \frac{97x^5 + 486x^4 + 491x^3 - 474x^2 - 576x}{24}$$

Simplifying,

$$3x^7 + 49x^6 + 315x^5 + 1015x^4 + 1722x^3 + 1456x^2 + 480x$$

$$= 20370x^5 + 10260x^4 + 103110x^3 - 00540x^2 - 120960x.$$

Subtracting $210x^4 + 1260x^3 + 2310x^2 + 1260x$ from each side, leaves $3x^7 + 49x^6 + 315x^5 + 805x^4 + 462x^3 - 854x^2 - 780x = 20370x^5 + 101850x^4 + 101850x^3 - 101850x^2 - 122220x$. Then dividing both sides by $x^5 + 5x^4 + 5x^3 - 5x^2 - 6x$, the resulting equation is

$$3x^2 + 34x + 130 = 20370$$

or,

$$3x^2 + 34x = 20240$$

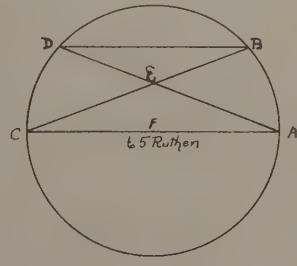
or,

$$3x^2 = 20240 \div\div 34.R.$$

17. ($= \frac{1}{2}$ of 34)
3
60720. + 289 ($= 17^2$)
289
√61009. | 247. ($= \sqrt{61009}$)
. . . | -17.
230 | $76\frac{2}{3}$.
dividing by 3

$76\frac{2}{3}$ is the value of x. It will be observed that this is the general method employed in this period. There is no indication that Wilken used negative roots.

JOHANN FAULHAUBER in Ingenieurs-Schul. (1637) applies algebraic principles to a familiar geometrical problem (p. 150).



Der Winkel ABC. Item CDA ist allwegen ein rechter Winckel in solchen Triangeln/da der Spitz B. oder D. in einem halben Eirckelden Umbkreiss anrührt/darauff procedier ich also:

65 quadriert 25
65. 25.
4225 □ 625 □
÷ 625
3600 ȝȯȯȯ|60 BC.

33.
33. auch quadriert.
1089 □. 4225.
1089.
3136.
6
ȝiȝȯ.|56. AD.
1ȯ.

Fersner setz ich BE 1A. und DE 1 Rx.

25. 1A. 56 ÷÷ 1. Rx.
25. 1A. 56 ÷÷ 1. Rx.
625. 1AA.

625. 1AA. gleich 3136. ÷÷ 112: Rx 1

Weiter.

33. 1. Rx. 60 ÷÷ 1A.
33. 1. Rx. 60 ÷÷ 1A.
1089. 1. ȝ ---gleich--- 3600 ÷÷ 120A 1AA.

Diese beyde Vergleichungen subtrahier ich von einander.

3600. ÷÷ 120.A. + 1.AA. gleich 1089. + 1ȝ.
* ÷ 625.---------- + 1.AA. gleich 3136. + 1ȝ. ÷ 112 Rx.
Rest. 2975. ÷ 120.A.----------gleich ÷ 2047.-----+ 112 Rx.
2047.
5022. ÷ 112 Rx. gl. 120A.
oder 60.A. gleich 2511 ÷ 56.Rx.

Facit 1.A. thut 2511. ÷ 56 Rx.
60.

quadriert 2511. ÷ 56.16305121. ÷ 281232.Rx. + 3136ȝ.
60. 3600. + 625

6305121. ÷ 281232. Rx. + 3136ȝ.|3600 625.gl.3136. ÷ 112.R. + 1.ȝ
403200. -----625
121968. Rx.- 2511. ÷ 112.R. + 1.ȝ
gleich ----------mit 3600
9039600. ÷ 403200.R. + 3600.ȝ
6305121. ÷ 3136
---2734479.---------- + 464ȝ

Oder.

464.ȝ. gleich 121968. Rx. ÷ 2734479.

Facit 1.Rx. thut $24\frac{3}{4}$. das ist DE.

2511. ÷ 56.Rx. thut 1.A.
60.

2511. 56
1386 $24\frac{3}{4}$
1125. $|18\frac{3}{4}$. 1386.

Ruth. thut 1.A. das ist BE.

60.

60.
$\div 18\frac{3}{4}$ 56.
CE. thut $41\frac{1}{4}$. Ruthen. $\div 24\frac{3}{4}$
$31\frac{1}{4}$. Ruthen thut AE.

Das vbrig ist leicht zufinden/sc.

This problem is significant in the use that Faulhauber makes of simultaneous quadratic equations. The equations preceded by an asterisk can be written in modern notation thus:

$$3600 + 120y + y^2 = 1089 + x^2$$
$$625 + y^2 = 3136 + x^2 - 112x$$

The study of the Third Decade of Mathematical exercises of John Camillus Gloriosus in which are contained various theorems and problems proposed to him for solution or noticed by him in his readings, (1639) leads to the observation that these readings, together with his correspondence led GLORIOSI to an unusual acquaintance with Vieta, for he goes on to relate (p.5) that

Franciscus Vieta in the first part of his Isagoges which he wrote on artem Analyticem, enumerates the books or treatises which are contained in a work restored by him, Mathematicae Analyseos or new Algebra; among them he mentions former and later notes to Logisticem speciosam. Out of curiosity I wrote to Marianus Ghetaldus (in 1619 the year when Mathematicum Algebram was published at Padua) asking if he had ever seen these Notes, since he knew Vieta at Paris and had enjoyed familiar association with him. Ghetaldus wrote me that while he had the former Notes, he did not have the later, nor had he ever seen them; indeed he suspected that Vieta had combined in his little work the former and the latter notes, which he gladly sent me, for the trustworthiness of these I am permitted to place here the letter of Ghetaldus.

Exemplar literarum Ghetaldi

Mando a v.s. quella operatta de Vieta, coie Notae priores ad Logisticum, questo e quanto ho io del suo, haueuo ancora de Recognitione aequationem,ma quella opera e stampata, v.s. non se marauigli se fin hora ho tardato, perche lo stentato un pezzo a trouar chi leggesse il caracterre francese, e poi trouatolo ha fatto mille errori mel copiarle, perche non solacio ho voluto mandarle l'originale, per non mandarle la copia cosi malamente scritta.

L'altra opera intitolata dal Vieta Notae posteriores, io non ho, ne credo che se ne trovi, e mi pare, se ben mi recordo che dit utte due ha fatto una, cioe questa che sono Notae priores, e per sine me le raccomando di Raguya alli 10 di November, 1620.

This little book which Ghetaldus sent me contained 46 propositions. And when everything was ready for it to be given to public print, word came to me that the aforesaid book had been published in France; I sought this diligently and at last received it through the effort and interest of a learned young man, a student of Mathematics, John Bire. This book, together with the Isagoge on the Analytical Art by the same author was printed at Paris in 1631 with the "scholijs" of a very learned young man I. de Beaugrand.....

Following the authority of Ghetaldus who wrote me that he suspected Vieta had combined in his little book the former and later notes, which perhaps Ghetaldus understood from Vieta himself, it is incredible that Vieta, if he had the former notes separate from the latter, would not have sent them to Ghetaldus.

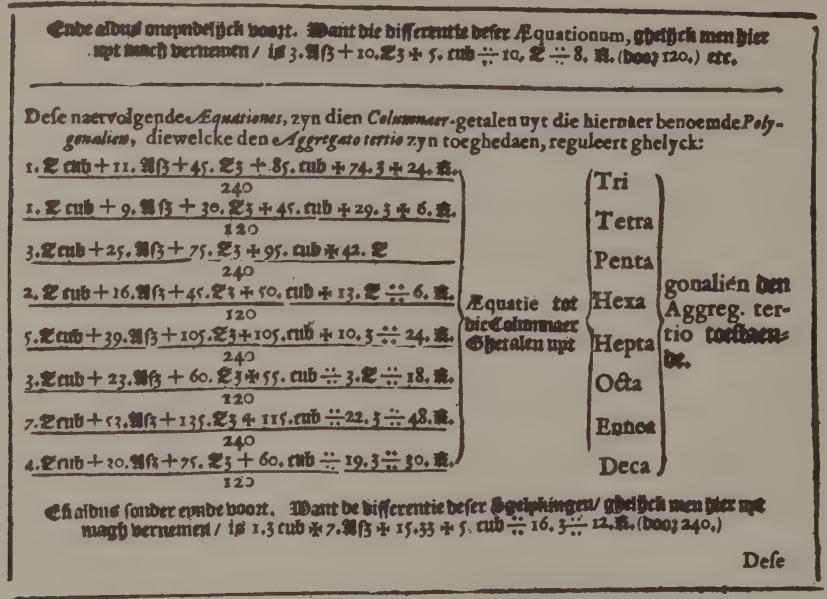
Later on he cites problems, especially those dealing with the cubic equations, from Diophantus, Vieta, Girard, Nonius, Coignet, Cardan, and Bombelli. Thus he demonstrates his wide knowledge of the outstanding authors of his day.

Under "De Aequationibus Quadraticis" (p. 82) he demonstrates the three types so general in this period. Example I proposes

1 Q aequari 8N + 20

To the square of half of the number is added an absolute number, & from the sum, the side is taken to which is added the half of the number N. This sum will represent the value of the side or 1N.

4	one half the number N	
16	its square)	add.
20	Number ?)	
36	Sum	
6	the side ?)	add.
4	half the number)	
10	value of the side or 1N	



Ende aldus onmyndelijck voort. Want die differentie deser Æquationum, ghelijck men hier uyt mach vernemen / is 3.Aſz + 10.Z3 + 5. cub ÷ 10. Z ÷ 8. R. (door 120.) etc.

Deſe naervolgende *Æquationes*, zyn dien *Columnaer*-getalen uyt die hiervaer benoemde *Polygonalien*, diewelcke den *Aggregato tertio* zyn toeghedaen, reguleert ghelyck:

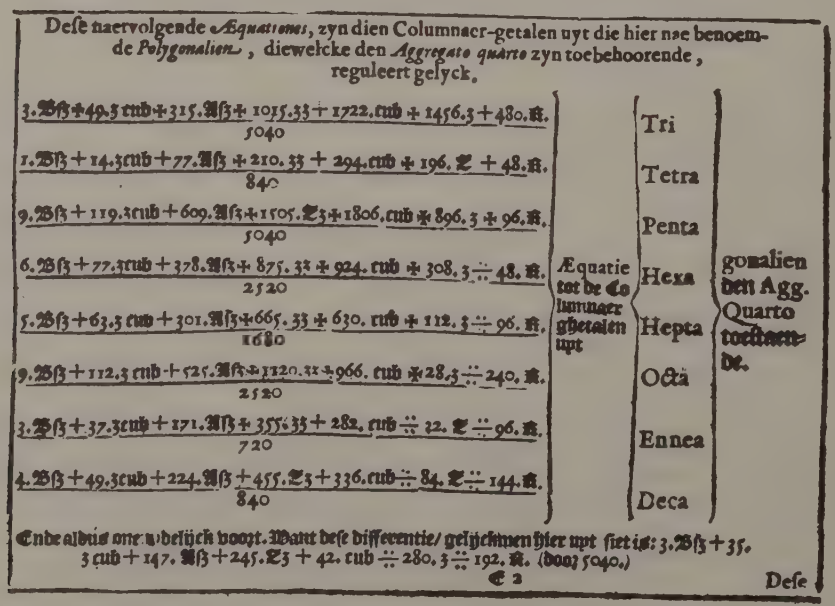
1. Z cub + 11. Aſz + 45. Z3 + 85. cub + 74. 3 + 24. R. / 240
1. Z cub + 9. Aſz + 30. Z3 + 45. cub + 29. 3 + 6. R. / 120
3. Z cub + 25. Aſz + 75. Z3 + 95. cub + 42. Z / 240
2. Z cub + 16. Aſz + 45. Z3 + 50. cub + 13. Z ÷ 6. R. / 120
5. Z cub + 39. Aſz + 105. Z3 + 105. cub + 10. 3 ÷ 24. R. / 240
3. Z cub + 23. Aſz + 60. Z3 + 55. cub ÷ 3. Z ÷ 18. R. / 120
7. Z cub + 53. Aſz + 135. Z3 + 115. cub ÷ 22. 3 ÷ 48. R. / 240
4. Z cub + 30. Aſz + 75. Z3 + 60. cub ÷ 19. 3 ÷ 30. R. / 120

Æquatie tot die Columnaer Ghetalen uyt: Tri, Tetra, Penta, Hexa, Hepta, Octa, Ennea, Deca — gonalien den Aggreg. tertio toebaende.

Eñ aldus sonder eynde voort. Want de differentie deser Ordelijnghen / ghelijck men hier uyt magh vernemen / is 1.3 cub + 7.Aſz + 15.33 + 5. cub ÷ 16. 3 ÷ 12. R. (door 240.)

Deſe

Fig. 42. From Wilken's Officina Algebrae (1636). This exhibits two of the many tables upon which Wilken based the formation of equations.



Deſe naervolgende *Æquationes*, zyn dien Columnaer-getalen uyt die hier nae benoemde *Polygonalien*, diewelcke den *Aggregato quarto* zyn toebehoorende, reguleert gelyck.

3. Bſz + 49. 3 cub + 315. Aſz + 1015. 33 + 1722. cub + 1456. 3 + 480. R. / 5040
1. Bſz + 14. 3 cub + 77. Aſz + 210. 33 + 294. cub + 196. Z + 48. R. / 840
9. Bſz + 119. 3 cub + 609. Aſz + 1505. Z3 + 1806. cub + 896. 3 + 96. R. / 5040
6. Bſz + 77. 3 cub + 378. Aſz + 875. 33 + 924. cub + 308. 3 ÷ 48. R. / 2520
5. Bſz + 63. 3 cub + 301. Aſz + 665. 33 + 630. cub + 112. 3 ÷ 96. R. / 1680
9. Bſz + 112. 3 cub + 525. Aſz + 1120. 33 + 966. cub + 28. 3 ÷ 240. R. / 2520
3. Bſz + 37. 3 cub + 171. Aſz + 355. 33 + 282. cub ÷ 32. Z ÷ 96. R. / 720
4. Bſz + 49. 3 cub + 224. Aſz + 455. Z3 + 336. cub ÷ 84. Z ÷ 144. R. / 840

Æquatie tot de Columnaer ghetalen uyt: Tri, Tetra, Penta, Hexa, Hepta, Octa, Ennea, Deca — gonalien den Agg. Quarto toebaende.

Ende aldus onmyndelijck voort. Want deſe differentie/ gelijckmen hier uyt ſiet is: 3. Bſz + 35. 3 cub + 147. Aſz + 245. Z3 + 42. cub ÷ 280. 3 ÷ 192. R. (door 5040.)

C 2

Deſe

This is similar to one he borrowed from Nonius.

CAJORI on p. 218 of his Notations recounts an illuminating letter which John Collins wrote to Wallis in which the former takes exception to Wallis' partiality to the works of Harriot and Oughtred. COLLINS says:

> You do not like those words of Vieta in his theorems, ex adjunctione plano solidi, plus quadrato quadrati, etc., and think Mr. Oughtred the first that abridged those expressions by symbols; but I dissent, and tell you 'twas done before by Cataldus, Geysius, and Camillus Gloriosus, who in his first decade of exercises, (not the first tract,) printed in Naples in 1627, which was four years before the first edition of the Clavis, proposeth this equation just as I here give it you, viz., lccc + 16qcc + 4lqqc - 2304qc - 1834qc - 133000qq - 54505c + 3728q + 8064N aequatur 4608, finds N or a root of it to be 24, and composeth the whole out of it for proof, just in Mr. Oughtred's symbols and methods. Cataldus on Vieta came out fifteen years before, and I cannot quote that, as not having it by me ... and as for Mr. Oughtred's method for symbols, this I say to it; it may be proper for you as a commentator to follow it, but divers I know, men of inferior rank that have good skill in algebra, that neither use nor approve it... Is not A^5 sooner wrote than Aqc? Let A be 2, the cube of 2 is 8, which squared is 64: one of the questions between Magnet Grisio and Gloriosus is whether 64 = A_{cc} or A_{qc}. The Cartesian method tells you it is A^6, and decides the doubt.

The first work of this period in which both positive and negative roots were indicated by plus and minus is Algebra ofte Nieuwe Stel-Regel (1639) by STAMPIOEN under the section, General Rule of Nieuwe Stel-Regel.

In the problem which begins by letting 1 ② + 4 ① equal 60 ($x^2 + 4x = 60$), the author uses the method of the Arabs, which was generally employed at this time and which is illustrated by the following solution:[191]

1 ② + 4 ① ghelijck 60	$x^2 + 4x = 60$
$\frac{1}{2}$)2	$\frac{1}{2}$ of 4 = 2
2	$2^2 = 4$
4	4 + 60 = 64
60	$\sqrt{64} = \pm 8$
√64	x = 6, -10
8 ofte -8	
2 2	
1 ① ghelijck 6. ofte -10	

[191] Algebra, 1639, p. 44.

In the check of this problem, as well as in the usual explanation accompanying the reduction of the equation, Stampioen emphasizes the fact that the square root of 64 is +8 or -8 because either a +8 or a -8 squared gives 64.

Furthermore this author, in order to check the problem, substitutes the roots in the equation, obtaining an identity thus:

1 ① ghelijck 6	1 ① ghelijck - 10
1 ② + 4 ① ghelijck 60	1 ② + 4 ① ghelijck 60
36 24	100 - 40
60	60

There appears in Stampioen's methods and notations a similarity to those of Stevin. The fact that he quotes Ramus leads one to conclude that the latter must have exerted some influence on Stampioen. Although Bombelli and Clavius used negative roots, Stampioen appears to be the first to indicate them by the modern + and -. He is, furthermore, one of the first to check problems in the manner just presented.

DE BILLY, a Jesuit and a teacher of mathematics, published in 1643 a Latin edition of his Algebra, differing very little from his French edition of 1637. It is this French edition which Leybourn translated into English in 1659 and included in his Arithmetick under the title, Abridgment of the Precepts of Algebra, from which we quote the following geometrical application (p. 361), remembering that De Billy symbolized the unknown and its second power by R and Q, respectively.

> There is a Semicircle, whose Diameter is divided according to mean and extream ratio, on which there is raised a perpendicular produced to the circumference, and the lesser line which is drawn from the extremity of the diameter, to this point of the circumference is R Q 20-2 ($\sqrt{20} - 2$) the quantity of the diameter and its parts, and of this perpendicular is demanded.

De Billy assigned x to the smaller segment of the diameter, and the greater segment is to have the same length as the perpendicular, making the whole diameter equal to the sum of these two segments, namely ($\sqrt{20} - 2 + x$). After expansion ($\sqrt{20} - 2 + x$)x = $(\sqrt{20} - 2)^2$. The author proceeds in the usual manner by taking one half of the coefficient of x, or $1 - \sqrt{5}$, the square of which is $6 - 2\sqrt{5}$. This added to the

ALGEBRA
Ofte
NIEUWE STEL-REGEL,
Waer door alles ghevonden wordt inde
WIS-KONST, wat vindtbaer is.

Noyt voor desen bekendt.

Gevonden, ende beschreven
Door JOHAN STAMPIOEN d' Jonghe,
Mathematicus.
Residerende in 's GRAVEN-HAGE.

Matth: 10.
Want daer en is niet bedeckt, het welck niet en sal ontdeckt worden: ende verborgen, 't welck niet en sal gewaten worden.

't GRAVEN-HAGE,
Ghedruckt ten Huyse vanden Autheur.
in *Sphæra Mundi*, 1639.

Fig. 43. Two title pages of Stampioen, Algebra ofte Nieuwe Stel-Regel (Huyse, 1639).

$x^3 - 6xx + 3x + 6 \propto 0$, ende soo veel falder mede komen wanneer men $x^4 - 10v^3 + 27xx - 6x - 24 \propto 0$, divideert door $x - 4 \propto 0$.

Voorts wanneer p is $\propto 0$, dan komter $x^3 - bxx + qx + \frac{r}{b} \propto 0$.
$+bb$

Wanneer q is $\propto 0$, dan komter $x^3 + pxx - pbx + \frac{r}{b} \propto 0$.
$-b \quad +bb$

Wanneer s is $\propto 0$, soo is m mede $\propto 0$, dan heeft men
$xx + \frac{l}{m}x + q \propto 0$, ofte $xx + px + \frac{r}{b} \propto 0$
$-\frac{l}{m} \quad -sl \qquad -b$
$+$

Wanneer r en s is $\propto 0$, dan is l en m mede $\propto 0$, dan heeft men
$x + p \propto 0$, ofte $x + \frac{r}{b} \propto 0$, ende alsoo met anderen.
$-b$

Hoe men den wortel kan vinden, van een Teerlinghs-vergheljckinghe, door 't trecken van den Teerlinghs-wortel uyt een twee-namigh getal.

WAnneer men de Teerlinghs-vergheljckinghen wil ontbinden door 't trecken van een Teerlinghs-wortel uyt een twee-namigh ghetal, soo moet men eerst de tweede Term uyt de Vergheljckinghe weech doen, soodet een is, ende dan falder een komen van dese drie voorvallen:

$x^3 \propto * - px + q.$
$x^3 \propto * + px + q.$
$x^3 \propto * + px - q.$

Om door Multiplicatie, soo een Verghelijckinghe toe te stellen, soo ser men $x \propto y + z$, dit aen weder-zijden Cubic ghemultiplicert soo heeft men $x^3 \propto y^3 + 3zyy + 3zzy + z^3$, dan ghestelt

stelt x, inde plaets van $y + z$, so krijgtmen $x^3 \propto 3zyx + y^3 + z^3$, dese Verghelijckinghe vergheloecken mett 't tweede voorval, so heeft men dese twee Verghelijckinghen:

$+p \propto 3zy$, soo is $\frac{+p}{3z} \propto y$.
$+q \propto y^3 + z^3$, soo is $y^3 \propto q - z^3$.

Stelt $-\frac{+p}{3z}$ in de plaets van y, soo heeft men $-\frac{+p^3}{27z^3} \propto q - z^3$, of $+p^3 \propto 27qz^3 - 27z^6$, ende $z^6 - qz^3 + \frac{1}{27}p^3 \propto 0$, nu gestelt $v \propto z^3$, soo komter $vv - qv + \frac{1}{27}p^3 \propto 0$. Ende $v \propto z^3 \propto \frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}$, soo is $q - z^3 \propto y^3 \propto \frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}$.

Soo komt $z \propto \sqrt[3]{\text{C.}\ \frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$, ende
$y \propto \sqrt[3]{\text{C.}\ \frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$. So is $x + y$ of
$x \propto \sqrt[3]{\text{C.}\ \frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} + \sqrt[3]{\text{C.}\ \frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$,
voor 't tweede Voorval.

Wanneer men dan $-p$, stelt in de plaets van $+p$, soo komt voor 't eerste Voorval
$x \propto \sqrt[3]{\text{C.}\ \frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}} + \sqrt[3]{\text{C.}\ \frac{1}{2}q - \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$.

Ende wanneer men $-q$, steldt in de plaets van $+q$, soo komt voor 't derde Voorval:
$x \propto \sqrt[3]{\text{C.}\ -\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} + \sqrt[3]{\text{C.}\ -\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$.

Voorbeeldt op t eerste voorval, zijnde $x \propto * - 3x + 36$, soo is $x \propto \sqrt[3]{\text{C.}\ 18 + \sqrt{325}} + \sqrt[3]{\text{C.}\ 18 - \sqrt{325}}$, want $\frac{1}{2}q$ doet 18, $\frac{1}{4}qq$ doet 324, ende $\frac{1}{27}p^3$, dat is den Teerlingh van $\frac{1}{3}p$ doet 1. Den Cubic-wortel uyt yder vande tweenamige getrocken komt $x \propto 1\frac{1}{2} + \sqrt{3\frac{1}{4}} + 1\frac{1}{2} - \sqrt{3\frac{1}{4}}$, dat is $x \propto 3$.

Hier staet te bemercken datter in dit eerste voorval maer een ware wortel kan wesen, ende dat de twee valsche wortels niet dan ingebeelde zijn, van welck voorval is mede iets aengeroert fol. 47.

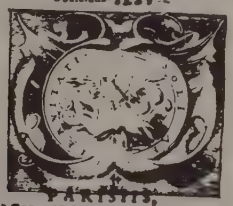
Fig. 44. Two pages showing Stampioen's solution of cubic equations in the above text.

NOVA
GEOMETRIÆ
CLAVIS
ALGEBRA.

CVIVS BENEFICIO APERITVR immensus Matheseos thesaurus, & resoluuntur plurima problemata hactenus non soluta in serie multarum quantitatum continuè proportionalium.

SIMVLQVE TRADITVR METHODVS vniuersalis, qua quilibet Marte proprio inuenire poterit innumera alia eiusmodi.

Authore P. IACOBO DE BILLY, Compendiensi Societatis IESV.



PARISIIS,
Apud MICHAELEM SOLY, viâ Iacobæa, sub signo Phœnicis.
M. DC. XLIII.
Cum Priuilegio & Permissu Superiorum.

Fig. 45. Title page of De Billy, Nova Geometriae Clavis Algebra (Paris, 1643). Below is a facsimile of two pages from the same text.

456 *Algebricarum præceptionum*

tiplicatus per seipsum, producit suum quadratum) 1 R autem multiplicata per 1 R, facit 1 Q per §. 3. capitis primi. Ergo $1R+1Q$ æquatur 20. Hic vides æquationem inter numerum datum 20, & numerum inuentum $1R+1Q$.

Similiter si quæratur numerus, à quo sublato vno triente & vno quadrante, residui quadratum æquetur numero inueniendo addito cum 3. Pono hunc numerum inueniendum esse 1 R, huius triens est $\frac{1R}{3}$ & quadrans est $\frac{1R}{4}$ quibus subductis ab 1 R, residuum est $\frac{5R}{12}$. Huius residui quadratum est $\frac{25Q}{144}$ quod iuxta quæstionis tenorem debet æquari numero inueniendo addito cum 15. Numerus autem inueniendus est 1 R. Ergo $1R+15$ æquantur $\frac{25Q}{144}$. Ecce æquationem inuentam.

§. 5. *Quomodo reducenda sit æquatio.*

AEquatio inuenta reducitur vel addendo eundem numerum vtrique termino æquationis, vel subtrahendo eundem numerum ab vtroque termino, vel sane multiplicando aut diuidendo vtrumque terminum æquationis per eundem numerum; ita enim semper manet illæsa & integra æquatio post eiusmodi operationes. vt si $1R+1Q$ æquatur 20, erit etiam æquatio inter $1R+1Q+2C$ & $20+2C$. Addendo videlicet 2C vtrique parti æquationis. Ita subtrahendo vtrobique 1 R, erit etiam æquatio inter $1Q$ & $20-1R$. Similiter multiplicando vel diuidendo vtrumque terminum æquationis per 3, habebis quoque æquationem inter $3R+3Q$ & 60 per multiplicationem: & per diuisionem deprehendes etiam æquationem inter $\frac{R+Q}{3}$ & $\frac{20}{3}$.

Epitome. 457

Porro, vt fiat hæc reductio vtiliter, ità procedendum est in ea, vt maior character cossicus maneat solitarius ex vna parte æquationis afficiens vnitatem: proindeque ex omnibus reductionibus quox factis, nulla est vtilis ad finem ab Algebra intentum præter secundam, in qua 1 Q reperitur solitarium ex vna parte & æquale 20 1 R.

Pari modo si cupias reducere æquationem alteram in §. 2. inuentam inter $1R+15$ & $\frac{1Q}{4}$ ita procedes: quoniam maior character cossicus est solitarius ex vna parte, laborabis tantum ad hoc vt afficiat vnitatem id autem præstabis hoc modo, reduces $1R+15$ in fractionem suppositâ vnitate & intellecta lineola: sic enim habebis $\frac{15+1R}{1}$ æquari $\frac{1Q}{144}$. Quoniam verò illæ duæ fractiones sunt æquales, erit talis ratio numeratoris primæ fractionis ad suum denominatorem 1 qualis est numeratoris secundæ fractionis ad suum denominatorem: igitur multiplicando eas fractiones decussatim, productis ex primo in quartum, æquabitur productò ex secundo in tertium, per Propos. 19. septimi Element.. Erit ita $144R+1872$ æquabuntur 25 Q. Ergo diuidendo vtrumque terminum æquationis illius posterioris per 25 numerum maioris characteris, erit æquatio inter $\frac{144R+1872}{25}$ & 1 Q. Vbi vides maiorem characterem cossicum ex vna parte solitarium afficere vnitatem. Hic est vnicus finis reductionis.

$$\frac{1R+15}{1} \times \frac{1Q}{144}$$

Dixi in regula Algebræ reducendam æquationem cum reductione opus est, quia contingit aliquando vt nulla sit necessitas: dum videlicet reperitur æquatio inter duos numeros cossicos simplices collaterales, nu-

absolute number or constant gives $30 - 10\sqrt{5}$ (in the original, 30 - R Q 500). The square root of $30 - 10\sqrt{5}$ is $5 - \sqrt{5}$, and this added to half of the coefficient of x makes $6 - 2\sqrt{5}$, or the value of the lesser part. Then 4 is the diameter.

De Billy shows no trace of the exponential notation, although he must have been familiar with the works of Descartes.

"The Doctrinae Analyticae Inventum Novum of Jacques de Billy is based on various letters sent to him from time to time by Pierre de Fermat"[192] and originally included at the beginning of the edition of 1670, of Bachet's Diophantus.

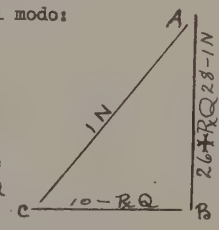
GRISIO quotes problems from Bombelli, Cardan, Bachet, Nonius, Peletier, Fra Luca, Ghaligai, Tartaglia, Lantz, Stifel, Stevin, Unicorno, and Gloriosi. Then, in the latter part of his text, he gives problems which he labels marginally, as for example,

L'Autor dell' Analisi si fa inuentore di simili estrazzioni.

These problems, which he claims as his own, are chiefly cubic and biquadratic. His general style is demonstrated by this problem from Lantz, which he incorporated into his text (p. 18)

Il Lantz ad libro quarto dell' instituzioni, a carte 165. al Problema 7. non scriue diuersamente de gl'altri, perche opera in tal modo:

" Esto (dic'egli) triangulum
" ABC, cuius basis BC fit 10
" - ℞2. 28. summa reliquorum
" 26 + ℞2 28 Pono AC. quod
" angulo recto opponitur esse
" 1 N: erit igitur AB 26 + ℞2
" 28 - 1 N. & quia quadratum
" lateris AC. est aequale
" laterum AB, BC. quadratis.
" Et est quadratum ipsius AC ℞2
" quadratum ipsius AB.
" 704 + ℞2 75712 - ℞2 1122
" - 52 N + 12. etc.



Base, or CB, $10 - \sqrt{28}$ AC = x

$CA + AB = 26 + \sqrt{28}$

$AB = 26 + \sqrt{28} - x$

$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$, etc.

Donde si scorge, che, hauendo multiplicato ℞ Q 28. per 1 N; hà prima quadrato 1 N; & fatto 1 Q; non 1 N; e chi quadrarà 26 + ℞ Q 26 - 1 N; troueŕa quel, ch'io dico.

The earliest edition of Arithmetica Practica by BASSI was that of 1654. On page 551 of the edition of 1666, which was the one examined, appears this typical problem.

Find a number which added to 42, is equal to the number squared.

Bassi assigns 1 cos. or x to the number, and x^2 he calls 1 quadrato, so that the equation becomes

1 quadrato è eguale ad 1 cos. piu 42

or $$x^2 = x + 42$$

By following the general plan of reduction, he arrives at 7 for the value of x, which he checks by substitution in the equation. Bassi writing as late as 1666, still adheres to the old notation of his predecessors, where cosa stands for x and quadrato for x^2.

In lieu of the edition of 1654 of Algebra by VAN DER HUIPS, his edition of 1661 was studied. There is a striking resemblance between this work and that of Stampioen. At the same time it is evident that no matter what his dependence on others may have been, his work like that of each of the Dutch authors, contains something which distinguishes him from the others. For example, van der Huips' classification of types of equations and general rules are those of Stampioen. Even the examples he uses, together with the manner of setting down the solution, are similar to those of Stampioen; yet Van der Huips differentiates his solution by the addition of an asterisk, the use of which he does not explain.

De helfte van't ① getall stelt in zijn Vier-kant, voeght daar toe het ledige, vanden √ treckt de heeft van't ① getall; komt 't begeerde.[193]

Tot verklaringh beziet 't volgende Voor-beelt.

1 ② = 16① - 60* $\qquad x^2 = 16x - 60$

[192] T. L. Heath, Diophantus of Alexandria, Cambridge, 1910, p. 28.
[193] Algebra, 1661, p. 127.

$\frac{1}{2}$	$\frac{1}{2}$ of 16 = 8
$\frac{1}{2}$ * 8	8 = 64
$\sqrt{\quad}$ 64	64 - 60 = 4
- 60	
$\sqrt{\quad}$ 4	$\sqrt{4} = \pm 2$
2 ofte -2	
* 8 * 8	8 $\pm$ 2 = 6 or 10
6 en -10	

De begeerdans

Van der Huips follows Ramus in the double use of the square root sign, designated by $\sqrt{}$, rather than by signs for the unknown.

Again van der Huips profits by the example set by Stampioen and accounts for both positive and negative roots, giving a proof of his solutions accompanied by appropriate explanations.

In some problems he employs x for the unknown, indicating a familiarity with the works of Descartes. For example, he gives the equation[194] $1x^2 = 4 - 12x$, which is solved as above, resulting in $\sqrt{40} - 6$ and $-\sqrt{40} - 6$ as the two roots.

Like his predecessors, he devotes the remainder of his text to cubic and biquadratic equations. The solution of these two types is based on the factor method, equating each factor to zero and thus arriving at the desired roots.

Van der Huips concludes this edition of his text by adding the first book of Diophantus.

COUTEREELS, a master of arithmetic, tells us on page 383 of his Arithmetica, that he bases his work on Tartaglia, Rudolff, Gemma Frisius, Forcadel, Ramus, Trenchant, Mellema, van de Gucht, Henrion, and van der Schuere. This applies, of course, chiefly to the arithmetical part of his text, but there is no doubt that these authors also influenced his algebraical work. His solutions are as straightforward as those of any of his fellow countrymen.

1♂ ghelijck	$\frac{50\rho}{25}$ - 600	$x^2 = 50x - 600$
	$\frac{25}{625}$	$\therefore x = 20,20$
add.	-600	
	$\sqrt{\quad}$ 25	
	+5 of -5	
add.	+25 25	
	30 20	

Facit 1ρ ghelijck 30 of 20.

Two roots are not given for the problem whose conditions determine the equation 1♂ + 1ρ ghelijck 72; 8 is the only answer given. The negative root is disregarded.

BUTEO, a monk of the order of Saint Anthony in Vienna, did not fail to exert an influence, as will be seen in the works of Visconti and Gosselin. In order to illustrate his solution of the class of equations under discussion, problem 25 on page 162 of his Logistica (1559) seems to involve a variety of interesting points. It reads,

Numerum quadratum inuenire, quo diusio in suum latus M 4, prouient 6 plus, quam ipsum latus.

Take such a number to be 1◇, whose side is 1ρ. Divide 1◇ by 1ρ M 4, giving $\frac{\diamond}{1\ M\ 4}$ 1ρ P 6.	x^2, the number
	x, the side
	$\frac{x^2}{x-4} = x + 6$
Make the equation by multiplying the contentum 1ρ P 6 by the denominator 1ρ M 4, making 1◇ + 2ρ M 24 [1◇, and 2ρ [24]. Divide by 2, giving 12, which is the "tetragonicum latus" of the number sought, namely 144.[195]	$x^2 = x + 6 \quad x - 4$
	$x^2 = x^2 + 2x - 24$
	$2x = 24$
	$x = 12$

I have here the formula of this equation

1◇ P 6ρ M 4ρ M 24 1◇

1ρ P 6 1◇

1 1ρ M 4

[194] Algebra, 1661, p. 126.

[195] The original had P 24, which is certainly a typographical error, since the author used M 24 in order to complete the solution.

which upon cross multiplication will give the same as is found in the explanation above.

Under the type, $ax^2 + bx = N$, which belongs to his first rule, it is desired to find a number whose square with its four sides and added to 7 makes 52. The equation thus becomes

1◇P 4ρP 7 [52

Subtracting 7 from both sides, one obtains

1◇P 4ρ [45].

$\frac{1}{2}$ of the numeri linearum, 4, is 2. Squaring 2 gives 4, which is to be added to the contenti numero or 45, making 49. The tetragonal side or square root is 7. $\frac{1}{2}$ of the lineal side, or 2, subtracted from 7 gives 5, the number sought.

Buteo next explains that if the constant happened to be 51 or 53, it would give a case which he labels as impossible. For then, proceeding as above, the equation would become 1◇P 4ρ[46, ($x^2 + 4x = 46$); and after taking $\frac{1}{2}$ of 4 and squaring, this result added to 46 would make 50, which does not have a "tetragonal side." To this he adds most forcibly "that there is no such number in the nature of things as is proposed here; that nothing more absurd could be done than this which cannot be explained adequately." He continues that a result like this is ridiculous, since it is not used in commerce nor applied to money. He continues that "these problems have been propounded partly by experience, partly by a vain show of subtlety or even by the base desire of obscuring the matter."

The resemblance between the work of Visconti and that of Buteo will be mentioned later in this work. Visconti does not, however, adopt Buteo's symbolism; rather, he uses that of Paciolo, although he might well have benefited by Buteo's symbol for equality.

Up to this point, the attempt has been made to present the study of equations in a somewhat chronological order. Beginning with Vieta, however, the sequence must be abandoned, since this great mathematician marks a sudden departure from the symbolism and general methods employed thus far in the period under consideration. Consequently, certain authors who published after 1630 have been placed in groups, and those who were undoubtedly influenced by the growing interest in analytic geometry have been placed in a separate group. This latter group includes Vieta, Harriot, Descartes, Renaldini, Herigon, Oughtred, Gibson, Wallis, Twysden, Rahn, and Moore. We find here the general employment of literal symbolism, with the use of small as well as capital letters of the alphabet. The authors recognized the double function of the + and - signs.

Vieta, the most eminent of French mathematicians of the sixteenth century, initiates this new era. Although some of his works were published late in life, the earliest being Isagoge in artem analyticam, edited first by himself at Tours, 1591, many were published after his death in 1603. This was true of De aequationum recognitione et emendatione libri duo, published by Alexander Anderson in Paris, 1615. The fact that mathematicians appreciated Vieta's contributions to mathematics even before his death is attested to by the correspondence between Ghetaldi and Gloriosi.

JOHN WALLIS, in Chapter XIV of his Treatise of Algebra, London, 1685, p. 64, speaks "of Francis Vieta and his Specious Arithmetick":

> After that Algebra had in such manner as is before said, been entertained, and cultivated in Europe, and had been so carried on as to reach all sorts of Quadratic Equations, and in good measure to those of Cubic Equations also; Franciscus Vieta, (about the year, 1590,) added a great improvement to it, by introducing what we call Specious Arithmetick; which gives Marks and Notes, not only to the quantities Unknown, but to the Known Quantities also; and exercises all the Operations of Arithmetick in such Notes and Marks as were before exercised in common Numeral figures.
>
> The name of Specious Arithmetick, is given to it (I presume) with respect to a sense wherein the Civilians use the word Species; for whereas it is usual with our Common Lawyers to put Cases in the name of John an-Oaks and John a-Stiles or John a-Down and the like, (by which names they mean any person indefinitely who may be so concern'd:) and of later times (for brevity sake) of J. O. and J. S. or J. D.; (or yet more shortly) of A, B, C, etc. In like manner, the Civilians make use of the Names of Titius and Sempronius, Caius and

Mevius (sic), or the like to represent indefinitely, any person in such circumstances. And cases so propounded they called Species. Now with respect hereunto, Vieta (accustomed to the Language of the Civil Law,) did give, I suppose the Name of Species to the Letters A, B, C, etc., made use of by him, to represent indefinitely any Number or Quantity, so circumstantiated as the occasion required. In like manner Euclide and others make use of α, β, γ, etc., to design Points, Lines, Angles, and other Quantities, to save the labour of describing them in long Periphrases or tedious Descriptions.

And to this way of Specious Arithmetick, Vieta doth accommodate, not only the ordinary Operations of common Arithmetick; but the Rules of Algebra before invented by former algebrists: differing from them herein, not so much in the substance of the Rules themselves (which are the same in the Old and New Algebra) as in a new Notation or Designation of those former Rules.

VIETA'S Opera Mathematica, 1646, employs the reduction of affected quadratics to pure quadratics, which is a method not employed formerly. In the following three formulas he gives both the new and the old notation.[196]

I

Si A quad. + B_2 in A, aequatur	$A^2 + 2AB = Z^2$
Z plano. A + B esto E. Igitur	$E = A + B$
E quad., aequabitur Z plano +B quad.	$E^2 = Z^2 + B^2$

Itaque, $\sqrt{Z \text{ plani} + B \text{ quad.}}$ -B fit A, de qua primum

quaerebatur sit B 1. Z planum 20. A 1 N.	$x^2 - 2x = 20$
1 Q + 2 N, aequatur 20, & fit N. $\sqrt{21} - 1$	$x = -1 + \sqrt{21}$

II

Si A quad. -B in A2, aequatur plano	$A^2 - 2AB = Z^2$
A - B esto. E. Igitur E. quadr.	$A - B = E$
aequatur Z plano B quad.	$A^2 - 2AB + B^2 = E^2$
	$E^2 = Z^2 + B^2$

Consectarium

Itaque + $\sqrt{Z \text{ plani} + B \text{ quad.}}$ + B fit A, de qua primum quaerebitur.	
Sit B 1. Z planum 20. A 1 N. 1Q - 2N,	$x^2 - 2x = 20$
aequabitur 20, & fit 1 N. $\sqrt{21} + 1$.	$x = 1 + \sqrt{21}$

III

Si D2 in A - A quad., aequetur	$2DA - A^2 = Z^2$
Z plano. D - E, vel D + E esto A.	$D - E = A$
Equad., aequabitur D quad. -	or $D + E = A$
Z plano.	$D^2 - Z = E^2$

Consectarium

Itaque, D minus, plusve

$\sqrt{D \text{ quad.} - Z \text{ plano}}$ fit A, de qua primum quaerebatur. Sit D 5. Z plano 20. A 1 N. 10 N - 1Q,	$10x - x^2 = 20$
aequatur 20. & fit N. $5 - \sqrt{5}$, vel $5 + \sqrt{5}$.	$x = 5 \pm \sqrt{5}$

Observe that two roots are not indicated in the first two cases, since the second root would be negative. In the last case, both are mentioned because both will be positive. This restriction prevented complete knowledge of the relationship which exists between the roots and the coefficients of the terms of a second degree equation.

Attention is called also to the fact that Vieta no longer used the vowels in solving numerical equations, but he resorted to the old notation, designating the unknown by N and its second power by Q, with the position of the coefficients reversed.

Besides his use of both types of symbols for the unknown, it should be observed that he now employs the vinculum in conjunction with the root sign to express a universal root. No doubt this addition was made by Van Schooten who edited the collected works of Vieta. Van Schooten, who also edited the works of Descartes, exhibits his influence in this addition to Vieta's symbolism. Vieta himself used the $\int$ of Ramus and Salignacus to express root extraction.

In the seventh book of HUME'S edition on De la correction des Equations are propositions similar to Prop. 1. Prob. 1.

Si Aij-Aj en Dj2, est egal à Zij * -Dij, la difference entre A & D sera egal à Zij*.	If $A^2 - 2AD = Z^2 - D^2$, $A - D = Z^2$

[196] Francisci Vietae Opera Mathematica, 1646.

Car la difference entre A & D, est A - D, & ou quarré sera Aij - Aj en Dj2, +Dij. Et parce que Aij - Aj en Dj*2, est egal à Zij* - Dij, adjoustez à tous les deux Dij, vous aurez Aij - Aj en Dj2 + Dij, quarré de la difference A - D egal à Zij*, ce qu'il falloit prouuer.

$(A - D)^2 = A^2 - 2AD + D^2$

because,

$A^2 - 2AD = Z^2 - D^2$.

By adding D^2 to both sides you have

$A^2 - 2AD + D^2 - Z^2$,

which is what we wish to prove.

We recognize this as Vieta's own problem expressed in new symbols. Hume completes the seventh book, the last book of his text, by problems of a similar nature leading to the cubic equation. Glaisher notes that "Vieta made very great advances by denoting the quantities forming the data of an equation by letters, but from an algebraical point of view his insistence on the equations being homogeneous, combined with his unwieldly notation for powers when written at length, rendered his operations very cumbrous."[197]

On page 166 of the Opera, Vieta gives a paradigm of the quadratic, which is explained in more detail by Renaldini and will be treated later.

In 1630 I. L. SIEUR DE VAV-LEZARD published at Paris his Introdvction en L'Art Analytic, ov Novvelle Algebre de Francois Vieta. On page 196 he gives one of the practical applications of VIETA'S method in Zetetiqve V.

Trouuer en nombre trois plans, lesquels ioincts deux à deux, comme aussi tous les trois diminuez d'vn plan donné, constituent vn quarré.

Find in a number three parts, such that joined two and two as well as all three, the three diminished by a part give a square.

Le plan donné soit Zp, la somme du premier & second soit Aq + Zp, afin que d'icelle estant osté Zp, reste le quarré de A. La somme du second & troisieme Aq + 2BA + Bq + Zp à cause qu'en soustrayant Zp d'icelle, le reste sera le quarré de A + B.

$A^2 + N$, sum of first and second.

$A^2 + N - N = A^2$

$A^2 + 2BA + B^2 + N$, sum of second and third.

$A^2 + 2BA + B^2 + N - N = (A + B)^2$

$A^2 + 2DA + D^2 + N$, the sum of all three.

$A^2 + 2DA + D^2 + N - N = (A + D)^2$

Finalement la somme des trois soit pour la mesme raison Aq + 2DA + Dq + Zp, afin que d'icelle estant osté Zp le reste soit le quarré de A + D. Donc si de le somme des trois en soustrait la somme du premier & du second, restera pour le troisiesme 2DA + Dq, & si de la mesme somme en osté celle du second & troisiesme, restera pour pour le premier 2DA + Dq - 2BA - Bq, & partant la somme du premier & troisiesme diminuée de Zp, sera 4DA + 2Dq - 2BA - Bq - Zp qui doit estre egale à quarré, soit icelluy Fq. donc $\frac{Fq + Bq + Zp - 2Dq}{4D - 2B}$ sera egal à A.

If from the sum of all three the sum of first and third are subtracted,

$A^2 + 2DA + D^2 + N - (A^2 + N) = 2DA + D^2$, or the third part.

$A^2 + 2DA + D^2 + N - (A^2 + 2BA + B^2 + N) = 2DA + D^2 - 2BA - B^2$, or the value of the first part. Taking the sum of the first and third from N, gives

$4DA + 2D^2 - 2BA - B^2 - N = F^2$. Then

$$A = \frac{F^2 + B^2 + N - 2D^2}{4D - 2B}.$$

Now if N is 3, B is 1, D is 2, F is 8, A will be 10. The sum of the first and second is 103, that is, $10^2 + 3$; the sum of the second and third is 124, or $11^2 + $; the sum of the three is 147, or $12^2 + 3$; finally the sum of the first and third is 67, or $8^2 + 3$. The first of the parts is, then, 23; the second, 80, and the third, 44.

In the Scholie which follows, he comments on Vasset, who also published an edition of Vieta in 1664, saying,

Vasset attribue ceste somme du premier & troisiesme plan, diminuée de Zp, au second plan, ce qui est contre le proposé, qui dit que cela doibt faire vn quarré, ce que demonstre le procedé pour paruenir à l'equation: mais ce qui l'a trompé c'est que ad aequandum quadrato, n'est pas dans le texte & n'estant bien scauant an ceste science, voyant qu'on parloit de la somme du premier & troisiesme, il l'a concluë egale au second plan.

CAJORI, after making A Revaluation of Harriot's Artis analyticae praxis,[198]

[197] Glaisher, Improvements, p. 128.

[198] Isis, Vol. XI, December, 1928, p. 316.

ARTIS ANALYTICAE
PRAXIS,

Ad æquationes Algebraicas nouâ, expeditâ, & generali methodo, resoluendas:

TRACTATVS

E posthumis THOMÆ HARRIOTI Philosophi ac Mathematici celeberrimi schedasmatis summâ fide & diligentiâ descriptus:

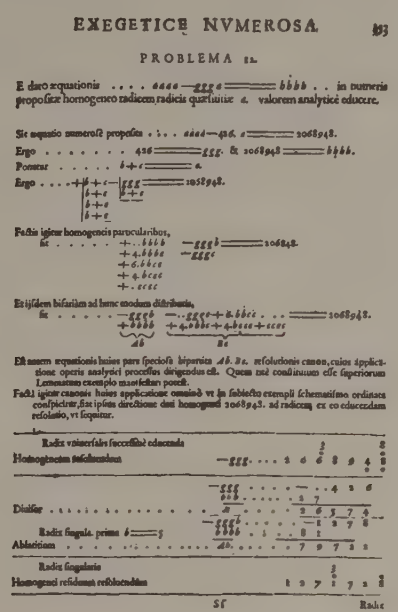
ET

ILLVSTRISSIMO DOMINO
DOM. HENRICO PERCIO,
NORTHVMBRIÆ COMITI,

Qui hæc primò, sub Patronatus & Munificentiæ suæ auspicijs ad proprios vius elucubrata, in communem Mathematicorum vtilitatem, denuò recusenda, describenda, & publicanda mandauit, meritissimi Honoris ergò Nuncupatus.

LONDINI
Apud ROBERTVM BARKER, Typographum Regium: Et Hæred. Io. BILL II.
Anno 1631.

Fig. 46. Title page of Thomas Harriot, Artis Analyticae Praxis (London, 1631).


EXEGETICÆ NVMEROSÆ 193

PROBLEMA 12.

Fig. 47. A page from the above-mentioned work of Harriot showing a partial solution of a biquadratic equation.

Johannis Wallisii, S.S. Th.D.
GEOMETRIÆ PROFESSORIS
SAVILIANI in Celeberrimâ
Academia OXONIENSI;
OPERVM
MATHEMATICORVM
Pars Prima,

Quæ Continentur,
Oratio Inauguralis.
Mathesis Universalis; sive, Arithmeticum opus Integrum, tum Numerosam Arithmeticam tum Speciosam complectens.
Adversus Meibomii, de Proportionibus Dialogum, Tractatus Elencticus.

OXONII,
Typis LEON. LICHFIELD Academiæ Typographi,
Impensis THO. ROBINSON. *Anno* 1657.

Fig. 48. Title page of John Wallis, Operum Mathematicorum (London, 1657).

THE KEY
OF THE
MATHEMATICKS
New Forged and Filed:
Together with
A Treatise of the Resolution of all kinde of Affected Æquations in Numbers.
With the Rule of Compound Usury; And demonstration of the Rule of false Position.
And a most easie Art of delineating all manner of Plaine Sun-Dyalls. Geometrically taught
BY
WILL. OUGHTRED.

LONDON,
Printed by THO. HARPER, for RICH. WHITAKER, and are to be sold at his shop in Pauls Church-yard 1647.

Fig. 49. Title page of Oughtred, The Key of the Mathematicks (London, 1647).

came to the conclusion that "the Praxis stands out as a remarkable treatise for the beginning of the century, making a further claim that it is a book which excels others in the great emphasis placed upon symbolic algebra. His notation became general with the subsequent modification (due to Descartes) that indices were used to denote the number of times a letter occurred."

The first part of the first section of Artis Analyticae Praxis contains a brief review of the fundamentals. On page 11 of the same section Harriot introduces the equation by

AEquationum irregularium ad formam legitimam reductiones exemplicatae:

giving those methods of reduction explained earlier in this work.

On page 16 is found "Canonicarum quadratici ordinis derivatio." Two propositions follow, the first of which will be considered here.

Aequatio canonica aa-ba
originali +ca = +bcab

a-b | ========== aa-ba
a+c | ca-bc posito

b. ipsi a. aequali deducitur.
Nam si ponatur a==b. erit a—b==0
Posito igitur a==b. erit a-b | ==0
a+c |
Est autem ex genesi a-b | ======= aa-ba
a+c | ca-bc

quae est aequatio originalis hic designata.

Ergo-----aa-ba
+ca-bc==0.

Ergo-----aa-ba
+ca ==== +bc quae est aequatio proposita.

Here he places a = b, later writing a - b = 0. Multiplying both sides by a + c he obtains aa - ba + ca - bc = 0, and therefore, aa - ba + ca = bc, which can be written as

$$x^2 + (-b + c)x = bc$$

On page 29 in the third section, further influence of Vieta is evident in the reduction of an affected to a pure quadratic. The methods are identical.

Harriot begins with the equation

aa - ba
+ ca = +bc

which he wishes to reduce to the pure quadratic aa = bb. This is accomplished by setting c equal to b. On page 52 of his Praxis, under proposition 2, he finds the roots of the equation

aa - ba
+ ca = +bc

to be b or c; for he goes on to prove that, if a is replaced by b, this equation becomes

bb - bb
cb = -bc

The same is true for c. In a lemma he proves that if d, say, which is not equal to b or c, is substituted in the equation, quite a contrary result will be obtained. Now let d = a; then the equation becomes

dd - bd
- cd = -bc

and dd - cd = bd - bc. If a common factor is taken from each member, expressed by Harriot as,

d - c | ==== d - c |
d | d |

then d = b. This is contrary to the hypothesis.

The application of equations is demonstrated in the Exegetice Numerosa, p. 117. Although the process is long, it is, nevertheless, very interesting from the standpoint of the influence it exerted on later authors, such as Gibson. Harriot, like Vieta, expresses a preference for pure quadratics.

E data aequationis quadraticae simplicis... aa-ff. in numeris propositae homogeneo radicem radicis quaesititiae a. valorem Analytice reducere.

Sit aequatio numerose proposita aa==48233025
Vnde----------48233025==ff.
Ponatur----------b + c==a
Ergo----------b + c | == 48233025.
b + c |

Factis igitur & ordinatis vt oportet homogeneis particularibus,

fit	+ bb	+ 2.bc	48233025
	Ab	+ ..cc	
		Bc	

/ Est autem aequationis huius pars speciosa bipartita Ab. Bc resolutionis Canon cuius applicatione operis analytici processus dirigendus est, quem rite constitutum esse ex sequenti Lemmatae constabit.

Facta igitur Canonis huius applicatione omnino vt in subiecto exempli schematismo ordinata conspicitur, fiat ipsius directione dati homogenei 48233025. ad radicem ex eo educendum resolutio vt sequitur.

Radix vniuersalis successive educenda	6 9 4 5
Homogeneum resoluendum	ff——48233025

Divisor	b	6
Rad. sing. prima	b = 6	
Ablatitium	bb	36

Radix singularis	6 . . .
Homogeneum reliquum resoluendum	12233025

Radix singularis	6 . . .
Homogenei reliquum resoluendum	42233025

Divisor	2.b	1 2 0
Radix sing. decuplata b = 60	2.bc	1 0 8 0
rad. sing. secunda c = 9	cc	81
Ablatitium	Bc	1 1 6 1

Radix aucta	6 9 . .
Homogenei reliquum resoluendum	623025

Divisor	2.b	1 3 8 0
Rad. aucta decupl. b = 690	2.bc	5 5 2 0
Rad. sing. tertia c = 4	cc	8 1
Ablatitium	Bc	5 5 3 6

Radix aucta	6 9 4 .
Homogenei reliquum resoluendum	69425

Divisor	2.b	13880
Rad. aucta decupl. b = 6940	2.bc	69400
Rad. sing. quarta c = 5	cc	25
Ablatitium	Bc	69425

Rad. vniversalis complete educta	6 9 4 5
	.....

Edatio igitur homogeneo 48233025. facta ipsius ad hunc modum resolutione educta est radix 6945. radici quaesititiae a. aequalis quae educenda erat.

In a similar manner, he finds the value of a in the equation $aa + da = ff$ by letting d = 432 and ff = 13584208 thus making the equation reduce to

$$aa + 432a = 13584208.$$

Proceeding exactly in the same manner as in the problem above, he continues as follows:

Radix vniuersalis successive educendo	3 4 7 6
Homogeneum resoluendum	ff——13584208

Radix singularis	b = 3, d	0432
	b	3
Divisor	A	3432
	db	1296
Rad. sing. prima	b = 3, bb	9
Ablatitium	Ab	10296

	d	432
Rad. sing. decuplata	b = 30 2b	60
Divisor	B	6432
	dc	1728
Rad. sing. secunda	c = 4 2.bc	240
	cc	16
Ablatitium	Bc	27328

Rad. aucta	3 4 . .
Homogenei reliquum resoluendum	5 5 5 4 0 8

	d	432
Rad. aucta decupl.	b= 340 2.b	680
Divisor	B	7332
	dc	3024
Rad. sing. tertia	c = 7 2.bc	4760
	cc	49
Ablatitium	Bc	51114

Radix aucta	3 4 7 .
Homogenei reliquum resoluendum	4 4 2 6 8

	d	432
Rad. aucta decupl.	b= 3470 2.b	6940
Divisor	B	7372
	dc	2592
Rad. sing. quarta	c = 6 2.bc	41640
	cc	36
Ablatitium	Bc	44268

Radix vniuersalis complete educta	3 4 7 6
Homogenei reliquum finale	.....

Edatio igitur homogeneo 13584208. facta ipsius ad hunc modum resolutione educta est radix 3476. radici quaesititiae a aequalis, quae ex intento Problematis educenda erat.

In estimating this book it must be remembered that it was written at least ten years before it was published, for Harriot died in 1621. The greater part of the work was devoted to cubic and biquadratic equations.

Although many historians have conjectured otherwise, Cajori proves conclusively[199] that Harriot rejected negative roots, adding that in this rejection Vieta and Harriot held less advanced views than did some of their predecessors, for example, Chuquet and Cardan. We have seen also that Harriot gives a general solution for quadratics. Though in special cases Harriot found the relation of the two roots to their sum and their product, yet nowhere is there found a general rule to express this relationship. As Cajori states,

"The Clavis mathematicae of OUGHTRED in its first edition of 1631 was a booklet of only 88 small pages. Yet it contained in very condensed form the essentials of arithmetic and algebra as known at that time."[200] Numerous minute alterations from the edition of 1631 occur in all parts of the books of 1647 and 1648. For further details as to changes of edition see Cajori in the work just referred to above (see f.n. 199).

In the preface to the edition of 1647 of the Clavis he says,

> Now my scope and intent in the first Edition of that my Key was, and in this New Filing, or rather forging of it, is, to reach out to the ingenious lovers of these Sciences, as it were Ariadnes tnread, to guide them through the intricate Labyrinth of these studies, and to direct them for the more easie and full understanding of the best and antientest Authors; such as Euclides, Archimedes, Apollonius Pergaeus that Great Geometer, Diophantus, Ptolomaeus, and the rest: That they may not only learn their propositions, which is the highest point of Art that most Students aime at; but also may perceive with what solertiousness, by what engines of aequations, Interpretations, Comparations, Reductions, and Disquisitions, those antient worthies have beautified, enlarged, and first found out this most excellent Science.

It will be noticed that Oughtred says that the Clavis was "to reach out to the ingenious lovers." Cajori in his article on The Works of William Oughtred, when commenting on the brevity of the edition of 1631, adds:[201]

> He (Oughtred) did not believe in conducting the reader along level paths and along slight inclines. He was a guide for mountain climbers and woe unto him who lacked nerve.

"Oughtred in his Clavis (1631)[202] retained Vieta's notation Aq, Ac, Aqq...for the powers of A, but he placed the numerical coefficient first instead of last, and omitted the word 'in' for multiplication: so that the square of A + E was written Aq + 2AE + Eq, ... and the expression $D^5 - 10D^2B^3 + 5D^4$ would have been written Dqc - 10DcBq + 5Dqq."

Oughtred does not use parentheses, but indicates aggregation by writing a colon (:) at both ends. Thus Q: A-E: means $(A-E)^2$ and √q: A+E: means $\sqrt{(A+E)}$.

Certain composite expressions are also represented in symbols, as for instance A + E = Z, A - E = x, where A > E. He also had a symbolism for $A^2 + E^2$, and so on.

As a specimen of Oughtred's style of writing, the following problem will illustrate what he meant when, in The Analytical Store, he wrote, "Which Analyticall furniture is no less precious then plentious."[203]

> Sunt duo numeri, quorum summa est Z: & maior ex ipsis ponitur A: quisnam est alter? quae ipsorum differentia? quod sub ipsis rectangulum? quae quadratorum summa? & differentia?

Using modern notation his answer is translated thus:

> The lesser will be Z - A. The difference 2A - Z. The rectangle (product) ZA - A^2. The square of the greater is A^2, the lesser, $Z^2 - 2ZA + A^2$. The sum of the squares is $Z^2 - 2ZA + 2A^2$ and difference, $2ZA - Z^2$.
>
> If in truth the lesser will be E, the greater will be Z - E; the difference, Z - 2E, the product, ZE - E^2; the square of the greater, Zq - 2ZE + Eq; and the difference, Zq - 2ZE.

His solution of quadratic equations, accompanied by a translation into English and modern mathematical symbols by Cajori[204] follows.

[199] Isis, Vol. XI, December, 1928, p. 318.
[200] F. Cajori, The Works of William Oughtred in Monist, Vol. XXV, 1915, p. 441--Cajori, Oughtred.
[201] Ibid., p. 449. [202] Glaisher, Improvements, p. 121. [203] The Key, 1647, p. 68.
[204] Oughtred, Clavis, 1631, p. 50. Cajori, Oughtred, p. 450.

Let $Z = A + E$ and $A > E$, $x = A - E$. From these relations we obtain identities as follows:

$$\tfrac{1}{4}Z^2 - AE = (\tfrac{1}{2}Z - E)^2 = \tfrac{1}{4}x^2$$

And if we know Z and AE, then from

$$A = \tfrac{1}{2}(Z + x)$$

$$E = \tfrac{1}{2}(Z - x)$$

we get $$A = \tfrac{1}{2}Z + \sqrt{\tfrac{1}{4}Z^2 - AE}$$

Having established these preliminaries, he then proceeds.

Given therefore an unequally divided line Z(10), and a rectangle beneath the segments AE(21) which is a gnomon. Half the difference of the segments $\frac{1}{2}x$ is given, and consequently the segment itself. For, if one of the two segments is placed equal to A, the other will be Z - A. Moreover, the rectangle is $ZA - A^2 = AE$. And because Z and AE are given, and there is $\frac{1}{4}Z^2 - AE = \frac{1}{4}x^2$ and by 5e.18, $\frac{1}{2}Z + \frac{1}{2}x = A$ and $\frac{1}{2}Z - \frac{1}{2}x = E$, the equation will be solved thus: $\frac{1}{2}Z \pm \sqrt{(\frac{1}{4}Z^2 - AE)} = A$, $\begin{cases}\text{major segment}\\ \text{minor segment}\end{cases}$.

And so an equation having been proposed in which three species (terms) are in equally ascending powers, the highest species, moreover being negative, the given magnitude which constitutes the middle species is the line to be bisected. And the given absolute magnitude to which it is equal is the rectangle beneath the unequal segments, without gnomon. As $ZA - A^2 = AE$, or in numbers, $10x - x^2 = 21$. And A or x is one of the two unequal segments. It may be found thus:

The half of the middle species is $\frac{Z}{2}(5)$; its square is $\frac{Z^2}{4}(25)$. From it subtract the absolute term AE(21) and $(\frac{Z}{4}) - A^2(4)$ will be the square of half the difference of the segments. The square root of this, $\sqrt{[(\frac{Z^2}{2})^2 - AE]}$ (2) is half the difference. If you add it to half the coefficient $\frac{Z}{2}(5)$, the longer segment is obtained, if you subtract it, the smaller segment is obtained. I say:

$$(\tfrac{Z}{2}) + [\sqrt{(\tfrac{Z^2}{4} - AE)}] = A\begin{cases}\text{major segment}\\ \text{minor segment.}\end{cases}$$

JOHN WALLIS, Oughtred's greatest pupil, devotes Chapter XV of his Treatise of Algebra, mentioned above, (pp. 67-69) to Oughtred and his Clavis, saying:

Mr. William Oughtred (our Country-man in his Clavis Mathematicae or Key of Mathematicks), first published in the Year 1631, follows Vieta (as he did Diophantus) in the use of the Cossick Denominations; omitting (as he had done) the names of Sursolids, and contenting himself with those of Square and Cube, and the Compounds of these.

But he doth abridge Vieta's Characters or Species, using only letters q, c, &c. which in Vieta are expressed (at length) by Quadrate, Cube, &c. For though when Vieta first introduced this way of Specious Arithmetick, it was more necessary (the thing being new) to express in words at length: Yet when the thing was once received in practice, Mr. Oughtred (who affected brevity, and to deliver what he taught as briefly as might be, and reduce all to a short view), contented himself with single Letters instead of Those words.

Thus what Vieta would have written

$$\frac{\text{A Quadrate, into B Cube,}}{\text{CDE Solid,}} \text{ Equal to F. G. Plane,}$$

would with him be thus expressed

$$\frac{A_q B_c}{CDE} = FG$$

And better to distinguish upon the first view, what quantities were Known, and what Unknown, he doth (usually) denote the Known by Consonants, and the Unknown by Vowels; as Vieta (for the same reason) had done before him.

He doth also (to very great advantage) make use of several Ligatures or Compendious Notes, to signify Summs, Differences, and Rectangles of several Quantities. As for instance, Of two Quantities A(the Greater) and E(the Lesser), the Sum he calls Z, the Difference X, the Rectangle AE———.

Mr. Oughtred in his Clavis, contents himself (for most part) with the solution of Quadratick Equations, and those of Higher Powers; having designed that Work for an Introduction into Algebra so far leaving the Discussion of Superior Equations for another work———. He contents himself likewise in Resolving Equations, to take notice of the Affirmative or Positive Roots; omitting the Negative or Ablative Roots, and such as are called Imaginary or Impossible Roots. And of those which he calls Ambiguous Equations, (as having more Affirmative Roots than one), he doth not (that I remember) any where take notice of more than Two Affirmative Roots: (Because in Quadratick Equations, which are those he handleth, there are indeed no more.) Whereas yet in Cubik Equations, there may be Three, and in those of Higher Powers, yet more. Which Vieta was well aware of, and mentioneth in some of his Writings; and which Mr. Oughtred could not be ignorant.

Smith in his History of Mathematics, Vol. I, page 375, says that

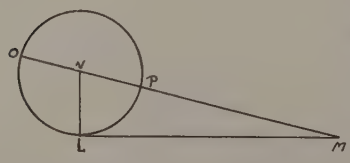
The fundamental idea in Descartes' mind was not the revolutionizing of geometry so much as it was the elucidating of algebra by means of geometric intuition and concepts; in a word, the graphic treatment of the equation. His imagination extended far beyond this, however, to the establishing of a universal mathematics in which algebra, geometry, and arithmetic should be closely related members. He began by extending the ancient idea of latitude and longitude, showing that any point in a plane is uniquely determined by two coordinates, x and y, the equation $F(x,y) = 0$ expressing a property which is true for every point of the curve. By studying the equation, therefore, he could, through the principle of a one-to-one correspondence, transfer his results at any time to the curve itself.

DESCARTES himself says,[205]

Thus all unknown quantities can be expressed in terms of a single quantity, whenever the problem can be constructed by means of circles and straight lines, or by conic sections, or even by some other curve of degree not greater than the third or fourth....

I shall therefore content myself with[206] the statement that if the student, in solving these equations, does not fail to make use of division wherever possible, he will surely reach the simplest terms to which the problem can be reduced.

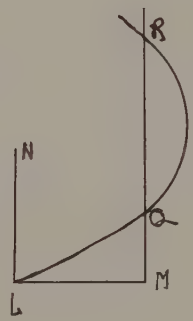
And if it can be solved by ordinary geometry, that is, by the use of str. lines and circles traced on a plane surface, when the last equation shall have been entirely solved there will remain at most only the square of the unknown quantity, equal to the product of its root by some known quantity, increased or diminished by some other quantity also known. (That is, an expression of the form $Z^2 - aZ \pm b$.) Then this root or unknown line can easily be found. For example, if I have $Z^2 - AZ + b^2$, I construct a right triangle NLM with one side LM, equal to b, the square root of the known quantity b^2, and the other side, LN, equal to $\frac{1}{2}$ a, that is, to half the other known quantity which was multiplied by z, which I suppose to be the unknown line. Then prolonging MN, the hypotenuse of this triangle, to O, so that NO is equal to NL, the whole line OM is the required line z. This is expressed in the following way:



$$Z = \tfrac{1}{2} a + \sqrt{\tfrac{1}{4} a^2 + b^2}$$

But I have $y^2 = -ay + b^2$, where y is the quantity whose value is desired. I construct the same right triangle NLM, and on the hypotenuse MN lay off NP equal to NL, and the remainder PM is y, the desired root. Thus I have

$$y = \tfrac{1}{2} a + \sqrt{\tfrac{1}{4} a^2 + b^2}$$



Finally, if I have $Z^2 = aZ - b^2$, I make NL equal to $\frac{1}{2}$ a and LM equal to b as before; then, instead of joining the points M and N, I draw MQR parallel to LN, and with N as a center describe a circle through L cutting MQR in the points Q and R; then Z, the line sought, is either MQ or MR, for in this case it can be expressed in two ways, namely,

$$Z = \tfrac{1}{2} a + \sqrt{\tfrac{1}{4} a^2 - b^2}$$

and

$$Z = \tfrac{1}{2} a - \sqrt{\tfrac{1}{4} a^2 - b^2}$$

Since $MR \cdot MQ = LM^2$, then if R = Z, we have

$$MQ = a - Z, \text{ and so}$$

$$Z(a - Z) = b^2 \text{ or } Z^2 = aZ - b^2$$

If, instead of this MQ = Z, then MR = a - Z, and again, $Z^2 = aZ - b^2$ Furthermore, letting O be the mid-point of QR,

$$MQ = OM - OQ = \tfrac{1}{2} a - \sqrt{\tfrac{1}{4} a^2 - b^2}$$

and

$$MR = MO + OR = \tfrac{1}{2} a + \sqrt{\tfrac{1}{4} a^2 - b^2}.$$

Descartes here gives both roots, since both are positive. If MR is tangent to the circle, that is, if $b = \frac{1}{2} a$, the roots will be equal; while if $b > \frac{1}{2} a$, the line MR will not meet the circle and both roots will be imaginary. Also, since $RM \cdot QM = LM^2$, $Z_1, Z_2 = b^2$, and $RM + QM = Z_1 + Z_2 = a$.

[205] Smith-Latham, The Geometry of René Descartes, Chicago, 1925, p. 10.

[206] Ibid., p. 13, 14.

In a footnote on page 17 of the work just quoted from, Smith comments:

It will be seen that Descartes considers only three types of the quadratic equation in z, namely, $z^2 + az - b^2 = 0$, $z^2 - az - b^2 = 0$, and $z^2 - az + b^2 = 0$. It thus appears that he has not been able to free himself from the old traditions to the extent of generalizing the meaning of the coefficients, - as negative and fractional as well as positive. He does not consider the type $z^2 + az + b^2 =$, because it has no positive roots.

KINCKHUYSEN, who wrote a geometry in 1660, gives a careful reproduction of Descartes in the Geometria ofte Meet-Konst, 1663, making such changes as serve to simplify the propositions given above. What is true of these propositions is true of his entire text, so that in the revision of statements and completion of figures, Kinckhuysen's text becomes a guide to Descartes.

"An example of the widespread influence of Descartes is given us by Conduit, a nephew of Newton, who said that Newton did not begin his geometry with Euclid but with Descartes."[207]

In the Supplement to Cursus Mathematici, HERIGONE include L'Isagoge de l'Algebra, 1642. In the midst of the solutions of many cubic equations, example 3, page 15, deals with a quadratic equation,

9 a2 ~ 72a	2|2	1456,	$9a^2 - 72a = 1456$	
9,	12,	16	9 12	16
1,	6,	91,	$a^2 - 6a = 91$	

En cet exemple, à cause qu'il faut reduire le nombre 9 de la puissance à l'vnité, et le faut diuiser par 9: mais à cause que le troisiesme nombre 1456 ne se peut diuiser par 9, on a pris 12 pour le diuiser du second: d'où s'ensuit que le troisiesme proportionel 16 est le diuiser de 1456. Partant la nouuelle equation sera e 2 ~ 12e 2|2 91, qui sont les quotiens des diuisions, en laquelle la valeur de la racine E est 13. Et parce que la proportion de 12 à 9 est $\frac{4}{3}$, en cet exemple, le commun diuiseur est $\frac{4}{3}$, par lequel on doit multiplier 13, valeur de la racine E, pour restituer la diminution faite par la diuision, & le produict sera $17\frac{1}{3}$ pour la valeur de la racine A.

In this example reduce the coefficient of the squared term to unity, by dividing by 9; but since the third number is not divisible by 9, one takes 12 as the divisor of the second term when the third proportion is 16, the divisor of 1456. According to the new equation $e^2 - 12e = 91$, which are the quotients of the divisors, the value of the root E is 13. And because the proportion between 12 and 9 is $\frac{4}{3}$ in this example, $\frac{4}{3}$ is a common divisor, by which one multiplies 13, the value of root E, in order to restore the diminution made by division. The product will be $17\frac{1}{3}$, the value of the root, A.

Question 2^{208}

Deux compagnies onct chacun pareil nombre d'escus à despartir: en l'vn il y a 4 hommes plus qu'en l'autre: partage faisant, il vient à chacun de la moindre compagnie 8 escus plus qu'à ceux de la plus grande: & tous les escus de chaque compagnie sont 172 plus que les hommes des deux compagnies: quel est le nombre de l'vne & de l'autre compagnie, & quel est le nombre des escus?

Cette question la 11 du 25 chap. de l'Algebre de Pelletier.

Translation of Question 2:

Each of two companies has a certain equal number of coins to divide; in the one there are 4 more men than in the other. In making the division there went to each member of the smaller company 8 coins more than to each member of the larger one: and all the coins of each company are 172 more than the number of men in the two companies. What is the number of men in each company and what is the number of coins?

Suppos.	a est le nombre de la moindre compagnie,
ergo	a + 4 est celuy de la plus grande, 2a + 176
hyp.	est le nombre des escus de chaque compagnie.
hyp.	$\frac{2a + 176}{a}$ 2|2 $\frac{2a + 176}{a + 4} + 8$,
isomer.	$2a2 + 176a$ / $8a + 704$ 2|2 $2a2 + 176a + 8a2 + 32a$,
antit.	$8a + 704$ 2|2 $8a2 + 32a$,
antit.	704 2|2 $8a2 + 24a$,
parab.	88 2|2 $a2 + 3a$,
	3 - 2 est la moitié de 3,
	9 - 4 est le quarré de $\frac{3}{2}$
	352 - 4 est le nombre donné reduit en quarts,
	361 - 4 est la somme,
	19 - 2 est la racine quarrée,
	3 - 2 est ladite moitié,
	16 - 2 est la reste,
	8 est le mesme reste reduit en entier.

[207] Brewster, Memoirs of Isaac Newton, Vol. I, p. 24.

[208] Herigon, Supplement, 1642, p. 96. Author's Translation.

Partant le moindre nombre de soldats sera 8, le plus grand 12, & la somme des escus de chaque compagnie 192.

This problem exhibits two characteristics. We see first a similarity of the terminology of Herigone to that of Girard and Vieta. The second and more important characteristic is the use of exponents. Although he was certainly familiar with the works of Girard and Vieta, and perhaps, through Girard, with those of Stevin, he does not adopt their symbolism. Herigone, with apparent confidence, expresses the positive integral exponents in Hindu-Arabic numerals, placing them to the right of the base rather than in the modern position. His use of 2|2 for equality illustrates his recognition of the importance of a convenient notation.

The method of simplifying equations found in RENALDINI, an erstwhile engineer following a teaching career as professor of philosophy and mathematics, has already been discussed. The addition of cross multiplication to his methods of[209] reduction is illustrated below.

Reductio aequationis inter minutias fit multiplicatione per crucem, & reducitur ad aequationem integrorum, Vt si esset equatio $\frac{122.}{4R} \rightleftharpoons 6$ 3R. fieri debet multiplicatio per

$$\frac{122.}{4R} \quad \frac{6 + 3R.}{1}$$

crucem vt fiat aequatio integrorum huius modi 24 R - 12 Q ⇌ 122.

These methods of reduction are followed by others explaining equations between $Q(x^2)$ and N and then between Q, R, and N.[210]

Varie, ac diuersae Methodus explicandi AEquationem inter Q + R & N. Methodus Diophanti.

In the margin the following paragraph is labeled "enarratio methodi Dioph."

5Q ✠ 10 R ⇌ 175	$5x^2 + 10x = 175$
5 Numerus ℚ)	5 " 5) multi-
175 Numerus) Multi-	175, the const.) plied
absolutus) plica.	875 product)
875 Productum)	25, $\frac{1}{2}$ of coef.) add
25 Quadr. dimidij) adde	of x squared)
numeri R.)	900 Sum
900 Aggregatum	
30 ℬℚ. Aggregati) Sub-	30, Sq.root of sum) Sub-
5 Dimid. numeri) trahe	5, half of coef.) tract
R)	of x)
25 Residuum diuidendum	25 Remainder, dividend
5 Numerus Quadratorum	5, number of x^2 and Di-
& Diuisor	visor
5 Quotiens, & 1 R.	5, Quotient, value of x
pretium	

Eadem pacto cum numeri surdi, siue Irrationales intercesserint procedendum est obseruata eorum lege.

In both the editions of 1644 and 1655 which are not essentially different, each type of equation is explained in a similar manner. After an illustration of these types each is succeeded by Methodus Communis Antiquorum, Methodus Petri Nonij, Methodus particularis Vietae, Methodus Stevini, Methodus Coigneti, Methodus Girardi, Methodus generalis Vietae.

Methodus Communis Antiquorum[211]

Exemplvm.

1Q + 10R ⇌ 96	$x^2 + 10x = 96$
5 Dimidium numeri R	$\frac{1}{2}$ of 10 = 5
25 Eius quadratum) adde	$5^2 = 25$
96 Num: absolutus)	96 + 25 = 121
121 Aggregatum	121 = 11
11 ℬℚ. Aggregati) Sub-	x = 11 - 6 or 5.
5 Dimid: num: R) trahe	

6 Lateris valor, sui 1 R. pertium.

Then follows a geometrical demonstration of the same problem.

[209] Opus Mathematicum, 1644, p. 56.
[210] Ibid., p. 65.
[211] Idem.

CAROLI
RENALDINII
PATRITII ANCONITANI,
ET IN CELEBERRIMA
PISANA ACADEMIA
Philosophiæ Ordinarij Interpretis.
OPVS MATHEMATICVM
IN QVO

Vtraque Algebra, vetus scilicet, & nova à se in Opere, hac de re prudenter edito, pertractata nouis præceptis; nouisq; demonstrationibus illustratur.

Methodus quoque Resolutionis, & Compositionis Mathematicæ longe copiosius, quàm ibidem, ad abstrusiora Theoremata, & Problemata enodanda declaratur.

Pars prior Numerosam Algebram complectens.

BONONIÆ, Ex Typographia H.H. Ducij. MDCLV.
Superiorum permissu.

Fig. 50. Title page of Renaldini, Opus Mathematicum (Bononiae, 1655).

150 *ALGEBRAE NVMEROSAE*

Et etiam Isomeriæ perficiendæ methodus altera. Reperiatur quinis communis diuiduus numerus omnium denominatorum; & multiplicetur numerator cuiuscunq; fractionis, per numerum, qui communem diuiduum metitur, per denominatorem numeratoris multiplicandi: integri autem numeri multiplicandi sunt per communem diuiduum. Exempligratia, sit æquatio huiusmodi, $\frac{60}{1} \text{ ⊕ } \frac{10}{1} \text{ ⊜ } \frac{200}{6}$. Communis diuiduus esto 30, vel alius numerus quicunq; quem possint omnes denominatores metiri. Numerus 2, metitur 30, per 15: proinde multiplicatis 60, per 15, producentur 900. Mox etiam videatur numerus, per quem numerus 5, metitur 30: & reperiemus 6: per hunc autem numerum multiplicentur 20, 1 produc untur 120. Deniq; 6, metiuntur 30, per 5: ob id multiplicat 1204, per 5, producentur 1020. Igitur æquatio illa superius posita per Isomeriam reducetur ad hanc $900 \text{ ⊕ } 120 \text{ ⊜ } 1020$.

Sit æquatio $1\text{R} \text{ ⊕ } \frac{17}{1} \text{ ⊜ } \frac{1\text{R}}{4} \text{ ⊕ } \frac{10}{1}$. Communis diuiduus esto 12, numerus, quem quidem 3, denominator fractionis $\frac{1}{4}$, metitur per 4: ducto autem 2, numeratore eiusdem fractionis in 4, producitur numerus 8. Præterea multiplicatis 17, per 4, numerum per quem 3, denominator fractionis metitur 12: & fit, 68. At verò 6, denominator fractionis $\frac{1}{4}$ R, metitur 12, per 2: ducto autem 5, numeratore eiusdem fractionis, in 2, producuntur 10. Itaque superior æquatio, per Isomeriam ad hanc reducetur. $12\text{R} \text{ ⊕ } 8 \text{ ⊜ } 10\text{R} \text{ ⊕ } 68$. & est 1R valor 30.

Sit æquatio supra iam allata $5\text{Q} \text{ ⊕ } 12\text{R} \text{ ⊜ } 185$: per diuisionem sit æquatio huiusmodi $1\text{Q} \text{ ⊕ } \frac{12\text{R}}{5} \text{ ⊜ } \frac{185}{5}$. Communis diuiduus esto numerus 10, quem denominator 5, metitur per 2: ducatur 1Q. in 10, fiunt 10Q: præterea ducantur 12R. in 2, numerum per quem denominator 5, metitur 10: & producentur 24R: ducantur 185, in 2,

CAPVT X. 151

in 2, & producentur 370: itaq; per Isomeriam reducta erit æquatio illa ad hanc $10\text{Q} \text{ ⊕ } 24\text{R} \text{ ⊜ } 370$; cuius radix est 5, quemadmodum, & illius $5\text{Q} \text{ ⊕ } 12\text{R} \text{ ⊜ } 185$, est etiam 1R, valor 5.

Sit æquatio superius posita $4\text{C} \text{ ⊕ } 6\text{Q} \text{ ⊜ } 10\text{R} \text{ ⊕ } 700$: diuisione instituta, fiet æquatio hoc modo $1\text{C} \text{ ⊕ } \frac{6\text{Q}}{4} \text{ ⊜ } \frac{10\text{R}}{4} \text{ ⊕ } \frac{700}{4}$. Communis diuiduus esto numerus 8, quem denominator 4, metitur per 2: multiplicemus 1C, per 8, & producentur 8C: multiplicemus 6, numeratorem fractionis $\frac{6\text{Q}}{4}$ per 2, & fiunt 12: multiplicemus 10R, per 2, & fient 20R: deniq; si multiplicentur 700, per 2, fient 1400: ob id per Isomeriam reducta erit æquatio illa, ad hanc $8\text{C} \text{ ⊕ } 12\text{Q} \text{ ⊜ } 20\text{R} \text{ ⊕ } 1400$. Huius autem æquationis radix est 5: nam 8C, valent 1000; & 12Q, valent 300, atq; demum 20R, valent 100: horum summa est 1400.

Isomeria fit etiam multiplicando singulos numeratores per denominatores aliarum fractionum.

Sit æquatio $\frac{6\text{R} \text{ ⊕ } 10}{1\text{R}} \text{ ⊜ } \frac{5\text{R} \text{ ⊕ } 45}{6}$. renouabitur ad hãc per Isomeriam, facta multiplicatione per crucem $24\text{R} \text{ ⊕ } 80 \text{ ⊜ } 30\text{R} \text{ ⊕ } 45\text{R}$: est autem 1R, pretium 10. Hoc autem quòd diximus, resolutione in numeros absolutos facilè constabit: nam si 1R, pretium est 10; ergo 6R, valebunt 60; igitur illa fractio $\frac{6\text{R} \text{ ⊕ } 10}{1\text{R}}$, erit 8: Insuper 8R, valor erit 80, ob id illius fractionis $\frac{8\text{R} \text{ ⊕ } 45}{6}$, pretium erit 8.

Si sit æquatio $1\text{C} \text{ ⊕ } \frac{3}{2}\text{Q} \text{ ⊜ } \frac{150}{2}$: per Isomeriam reducetur ad hanc $1\text{C} \text{ ⊕ } 3\text{Q} \text{ ⊜ } 1300$. Nam Q, distat à C, per R: ideo ducatur 3, numerator fractionis quadratorum in 2, numerum cuborum, per quem institutu fuit diuisio, & fit 6, quo diuiso per 2, denominatorem fractionis, fit iterum

Method of PETRI NONIJ[212]

1Q + 20R ⟸ 125	$x^2 + 20x = 125$
400. Quadratum	$400 = 20^2$
500. Quadruplum	$500 = 4 \cdot 125$
___ numeri absoluti.	___
900. Aggregatum	900 = sum
30. Aggregati latus) Sub-	$30 = \sqrt{900}$) Subt.
20. Numerus Radicum) trahe.	20 = coef-)
10. Residuum	ficient)
5. Residui dimidu-	10 = Difference
im, & 1R.	5 = 10 - 5,
valor.	the value
	of x

Particular Method of VIETA[213]

1Q - 30R ⟸ 400 $\qquad x^2 + 30x = 400$

This method is practically the same as that of Petri. Squaring 30 and adding 4 times 400 gives 2500, whose square root is 50. Two parts of 50 are taken as 40 and 10, without giving any further reason for this sum than that it is the sum and difference of the extremes. Therefore, the positive root will be 10.

Method of STEVIN[214]

1Q + 10R ⟸ 75 $\qquad x^2 + 10x = 75$

Stevin's method substitutes 1 for x, then 10x = 10 and 1^2 is added to 10 gives 11. There then results an equation between 11 and 75, which the author says is false. He makes a trial of 2 and 7, and in each case he finds that they fail to satisfy the equation. The number 2 substituted in the equation gives 24, which is less than 75; whereas a substitution of 7 yields 117, a number greater than 75. Therefore the number which will satisfy the equation lies between 2 and 7. He finds that 6 does not satisfy it; but upon making trial of 5, an identity results, which justifies the conclusion that 5 is the root.

Method of COIGNET[215]

1Q ⟷ 10R ⟸ 75 $\qquad x^2 + 10x = 75$

Coignet finds all the possible factors of 75 to be 1, 3, 5, 15, 25, 75. And, as in Vieta's method, the number that satisfies the equation is the desired root.

Method of GIRARD[216]

1Q ⟸ 10R ⟷ 75	$x^2 = 10x + 75$
1 ——— 75)	1 ——— 75)
3 ——— 25) Efficientes	3 ——— 25) Factors of 75
5 ——— 15) numeri 75.	5 ——— 15)

Girard also puts down all the possible factors of 75. He, then, divides the equation by x, and it becomes

$$x = 10 + \frac{75}{x}$$

Upon choosing one of the factors, as 15, such that 10 subtracted from it will be equal to 75 divided by that number, or 15, he shows that the other factors would not satisfy the equation.

Renaldini follows this by giving a paradigm of the general method of Vieta in which a quadratic equation with positive roots is analyzed.

The citation of the methods of different authors makes Renaldini's work valuable for purposes of comparison. It is here that we find interesting examples of Coignet's method rather than in his own texts. Although we have seen Girard's admiration for Stevin, it is not difficult to conclude that Coignet, too, drew his inspiration from the same source. Coignet edited Menher's Algebra, which bears further witness to the esteem in which the latter was held in his own country.

Renaldini used both the old and the new notation, showing partiality to the new, particularly in Artis Analytica Mathematum, parts I, II, 1669. In part II a peculiar symbol for the negative is used. On page 124 he says that if $ba \frown ca = df \frown ef$, then by "parabolismus"

$$\frac{df \frown ef}{b \frown c} = a.$$

The difference in these two symbols is the absence of the vertical extension of the

[212] Opus Mathematicum, 1644, p. 66.
[213] Idem.
[214] Opus Mathematicum, 1655, p. 183
[215] Ibid., p. 184
[216] Idem.

horizontal brace, since the letters used in the fraction were smaller and necessity dictated the abbreviation of the original brace.

JONAS MOORE, a teacher of mathematics, naively states, in the preface to his Arithmetick (1660), the purpose of the book when he writes,[217]

> In the second Book of Algebra I have not proceeded to further Mysteries, but contented myself to employ my whole strength to explain what was formerly delivered, in the best manner I could imagine, and doubt not but have attained my aim, To write nothing that might be superfluous, nothing too difficult or obscure: That Book serving for an introduction to Mr. Oughtred's Clavis Math. Vietta [sic], Des-Chartes, Diaphantus, and the rest of those learned Authors, that have proceeded in this worthy Art.

In introducing the equation he mentions that he uses material from Oughtred, Bachet and de Billy whenever occasion offers and indeed he does begin his work with equations very similar to those in Oughtred's Clavis, Chapter XVIII, The Analytical Store. Further on he states that a certain method is Mr. Harriot's.

OUGHTRED himself in the preface to The Key of Mathematicks, 1647, after giving credit to Mr. Ward and others for the production of this edition, adds:

> ...and lastly (but indeed chiefly) unto my knowing and worthy friend, Mr. Jonas Moore, who is singularly learned, painfull, and curious, as well in this way, as in other mathematicall Sciences...

MOORE'S method of the reduction of the equation is identical with that of Oughtred. He asks the question,

> Having two numbers given, whereof a the greater and b the lesser, to find their summe, difference, rectangle, proportion, summe of squares, and difference of squares, z, x, p, $\frac{r}{s}$, aa, bb, m & n. (2) D: a & z gr. b, x, $\frac{1}{s}$, paa, &c. AEquo (per 1) a + b = z & b = z - a.

a	a	$x = a - (z - a)$ or
b	z - a	$2a - z$
x	2a - z	$p = a(z - a) = za - a^2$
p	za - aa	
$\frac{r}{s}$	$\frac{zr - ra}{a}$	$r(\frac{b}{a}) = r(\frac{z-a}{a})$

aa	aa	$(z-a)^2 = z^2 - 2za + a^2$
bb	zz - 2za + aa	
m	zz - 2za + 2aa	$a^2 + b^2 = z^2 - 2za + 2a^2$
n	2za - zz	$a^2 - b^2 = 2za - z^2$

The reader's attention is called to the modern character of this solution.

Proposition 2 attests to his dependence on Oughtred, for it reads:

> If a line or number Z 12, be cut into two unequal parts A and E, 8 and 4, the Rectangle or multiplication made of the whole line or number Z, and of the Segments A and E, will be equal to the square of the whole line.
>
> 2.Z = ZA + ZE
> 144 = 96 + 48

On page 69 of the section on the equation he states three types:

> If the one side of the AEquation be an affected AEquation, and yet the Indexes thereof are in Arithmetical proportion, then it being some of these that follow, ZA - Aq = AE. Aq - XA = AE. Or Eq + XE = AE.

Here A is the Unica and E the gnomon.

The resolutions by Chap. 16, Section 9 (p. 70) are

$$1.\ \frac{Z}{2} \pm \sqrt{q:\ \frac{Zq - 4AE}{4}} = \frac{A}{E}$$

$$2.\ \sqrt{q:\ \frac{Xq + 4AE}{4}} : \mp \frac{X}{2} = \frac{A}{E}$$

Moore gives the regular rule for the reduction of an equation. The colon placed before and after the fraction signifies a universal root so that

$$\sqrt{q:\ \frac{Xq + 4AE}{4}} : \text{ means } \sqrt{\frac{X^2 + 4AE}{4}}.$$

It is seen that he also uses the $\pm$ to indicate the two roots. Although this is not the earliest use of the $\pm$, it is one of the first examples of the application of the double sign to the quadratic formula. It is, however, the first appearance of the reversal of the double sign revealed by this study.

The translation of the title of J. TWYSDEN'S work explains exactly what one

[217] Moore, Arithmetick, 1660, p. 72

can expect to find in Certain Mathematical Problems, (Concerning Triangles as well Oblique as Rectangled), Analytically Resolved, and Effected. Although this material rightly belongs to a subsequent discussion of geometrical demonstrations, it will be included here in order to illustrate the importance he attaches to the formula.[218]

Problem II

In a right-angled triangle p, the perpendicular, let fall from the right angle upon the hypotenuse, and b the difference of the segments of the hypotenuse are given to find the triangle.

Sit â minus segmentum, b + a erit majus, & b + a in a, hoc est, ba + aa ꝏ pp.

Ergo Canon. $\sqrt{\frac{1}{4}bb+pp:} \pm \frac{1}{2}b$ ꝏ a.

The probleme may be thus otherwise propounded. In three quantities in continual proportion, the middle term is given, and the difference of the extremes. To find the rest.

Geometrically thus. Upon the diameter EF, produced infinitely erect mI at right angles, equal to p the perpendicular given, and measure off mH, equal to $\frac{1}{2}$b, draw HI which shall be $\sqrt{\frac{1}{4}bb + pp}$ by the 47 of the I of Eucl. upon that as semidiameter describe a semicircle, and the analytical Canon is observed. For Em is $\sqrt{\frac{1}{4}bb + pp:} + \frac{1}{2}b$, and mF is $\sqrt{\frac{1}{4}bb + pp:} - \frac{1}{2}b$. For Em, mI.mF ÷÷ by reason of the similitude of the triangles EIm. mIF.

The other problems have a similar development, leading to a few solutions of the cubic and quartic equation.

Twysden also uses the double ± sign and adds Oughtred's sign of proportionality to his symbolism. His equality sign has the appearance of an inverted and reversed "ae," resembling Descartes' sign very closely.

RICHARD BALAM devoted Chapter XIV of his Algebra, 1653, to A Resolvend Equation, specious and numerous. An Equation pure and mixed. He goes on to define "A resolvend (sic) specious Equation" (p. 119) as one in which one of the two parts is a known magnitude taken to a power. A pure equation

> is when one onely power of the supposition, cleared from the coefficient in the Nome with it, is equal to an Absolute... As if 5A(2) = 45.

Then the pure Equation will be $A(2) = \frac{45}{5} = 9$. Then A = r9 = 3.

This may be written as $5A^2 = 45$, or $A^2 = 9$. Then $A = \sqrt{9}$ or 3. "Nome" means term.

Balam goes on to define,

> A mixt Equation (commonly called affected) is when the supposition, with a power of the supposition, or when severall powers of the supposition, (that is, when severall Cossicks of a Progression) either all affirmed, or one affirmed and another denyed, are equall to an absolute. As,
>
> A(2) + 2A = 15. A(3) - 3A(2) + 6A = 18.

Balam explains that the Highest Cossick of a mixed equation is always to be cleared of the coefficient. And on page 122 he tells us that

> L, or a root in an Equation, is equall, oftentimes ambiguously to as many severall numbers as there are Cossicks in the Equation.

Farther on (p. 123) he says,

> A mixt Equation may be exprest also by the Coefficients, and the due Cossick notes without L, or any Symboll before the notes. As, A(3) + 2A(2) = 9 may be represented by L(3) + BL(2) = P(3). Or by 1(3) + 2(2) = 9.

In the next chapter Balam treats the affected quadratic equation. In the first case (p. 126) he points out that

> ...the higher Cossick more by the lower is equall to the Absolute, and the lower Cossick is the root squaring of the square of halfe the Coefficient more by the Absolute, being lesse by halfe the Coefficient. As
>
> L(2) 2L = 3. Then
> L = r(2): (2):(1):(3): —1 = 1

which means that for $x^2 + 2x = 3$, $x = \sqrt{1^2 + 3} - 1 = 1$, where r(2) represents square root, the two colons indicate the radicand. (2) : 1 : means 2^2.

In the third case (p. 127) Balam gives the equation 4L - L(2) = 3, or $4x - x^2 = 3$, then he says that

[218] Problementum Quorundam Mathematicorum in Foster, Miscellanies, 1619, p. 4.

Syntaxis Mathematica:

Or, A

CONSTRVCTION

Of the harder Problemes of

GEOMETRY:

With so much of the

CONICKS

as is therefore requisite,

And other more ordinary and usefull *PROPOSITIONS* intermixed: And *TABLES* to several purposes.

The Contents *follow in the ensuing leaf.*

By *THO. GIBSON.*

Virgil.
Tempora dispensant usus & tempora cultus.

LONDON,
Printed by *R. & W. Leybourn*, for *Andrew Crook*,
and are to be sold at his Shop at the
Green Dragon in St. *Pauls*
Church-yard. 1655.

Fig. 52. Title page of Gibson's Syntaxis Mathematica (London, 1655).

(70)

Make $\frac{gg}{a} = f$, therefore $agg = afe$,

Secondly, make $ggg - agf = gg$, from thence it is plain that $bbc - agg = gge$, which was first to be done.

Secondly, to reduce $c^6 - b^6$ into one intire Solid, though not into a Squared Cube as d^6, as is usurped by *Mr. Harriot* for brevity in writing, or facility in reasoning, *Pag.* 100, supposing that done which cannot be done by streight lines and cirels hitherto.

Now therefore seeing $c^6 - b^6$ is produced by multiplication of $ccc + bbb$ into $ccc - bbb$,

Make $\frac{cc}{b} = f$, and $\frac{bb}{c} = g$, & $f + g = q$

and $f - g = p$, therefore $bcf = ccc$, and $bcg = bbb$, and $bcq = ccc + bbb$. Secondly also $bcp = ccc - bbb$: And therefore $bbccpq = cccccc - bbbbbb$, which was secondly to be done.

Example in Numbers.

Put $b = 2$ and $c = 3$: Then $ccc = 729$, and $bbbbbb = 64$, and then $cccccc - bbbbbb$, that is $729 - 64 = 665$, which is produced by multiplying $27 + 8$ by $27 - 8$, that is, 35 by 19. Now make $f = \frac{1}{2}$, and $g = \frac{1}{2}$

(71)

$g = \frac{1}{2}$, then $f + g = 3\frac{1}{2} = q$, and $f - g = 3\frac{1}{2} = p$. And $bcq = 35$, and $bcp = 19$. And lastly $bbccpq = 665$.

Moreover, if you make $pq = xx$, the Solid is further reduced to $bbccxx$, which although it be not a Squared Cube, yet it hath a square root, namely bcx, which may be of good use in many cases to resolve Æquations into Analogismes, of which kinde of Demonstration, by help of *Euclide* 6. 14. some notice is taken before in *Chap.* 2.

NOTE.

The three cases of the æquatio $a^3 - 3bba = 2c^3$, mentioned in the beginning of the fourth *Rule* of the last *Chap.* are called by *Mr. Harriot*, the first *Hyperbolicall*, the second *Parabolicall*, the third *Elipsicall*, because of some similitude between them and those sections, of which three Cases, the first is resoluble by a Conique Section, the second by a Circle, and the third not at all.

Multiplication and *Division* of Solids is altogether as easie as Addition or Subtraction, for if one would divide ccc by bb, make $\frac{ca}{b} = x$,

and again make $\frac{cx}{b} = z$, then z is the Quotient required.

Ex-

Fig. 53. Two pages from the above text.

Le = 2--- = 1
r(2) : (2) ⋮ 2 ⋮ -3:
La = 2 + = 3

or, $x_1 = 2 - \sqrt{2^2 - 3}$ or 1

and $x_2 = 2 + \sqrt{2^2 - 3}$ or 3.

It has been noted thus far that the rule for the solution of the equation resembles the earlier general method. Moreover, he employed the exponential symbolism peculiar to many authors, particularly that of Girard.

Balam also resorts to the methods of Harriot and Oughtred. On page 128 he gives the following explanation:

The Coefficient with the lower Cossick is in the first and second cases, the difference of the two unequall Factors and the Symboll of it is D. In the third case the summe of the unequall Factors and the symboll is S.

Therefore the composing and resolving of a mixt Equation of two, as 2 to 1, may be represented by Symbols.

1) In the first case thus, L(2) + DL = F. Then Le = r(2) : (2) ⋮ $\frac{D}{2}$ ⋮ + F: -- $\frac{D}{2}$, that is, if $x^2 + Dx + F$, then

$$x_1 = \sqrt{\left(\frac{D}{2}\right)^2 + F} - \frac{D}{2}$$

2) In the second case L(2) - DL = F then La = r(2) : (2) ⋮ $\frac{D}{2}$ ⋮ + F: + $\frac{D}{2}$ or

$$x_2 = \sqrt{\left(\frac{D}{2}\right)^2 + F} + \frac{D}{2}$$

3) In the third case thus, SL -- L(2) = F, then L = E = S -- L = A = S + r(2) : (2) ⋮ $\frac{S}{2}$ ⋮ -F: which result resembles the one obtained by using the quadratic formula where

$$x = s \pm \sqrt{\frac{s^2}{4} - F}.$$

We might note here that S = A + E and D = A - E.

The remaining twenty-eight pages deal with the subject in this same manner. One therefore may be justified in assigning Balam to the transitional period. In fact the word "transitional" seems to apply better to Balam than to any other author thus classified, since he has made a very definite connection between the old and the new methods.

GIBSON begins the study of equations in Chapter II, p. 25, of Syntaxis Mathematica (1655). After explaining that in the equation aa + bc = cc, b is called the Coefficient, his reference to old rules makes a neat transition between the old and the new.

Note also that the three sorts of mixed aequations above expressed are all that can happen to quadratiques: and by some one of these, all problems whatsoever transcending plaine Divisione and not reaching Solids, are to be resolved by finding the Root a, according to Old Rules.

In the First, aa + bc = cc

Unto the quantity given, namely cc, adde the Square of half the Coefficient, it makes cc + $\frac{bb}{4}$. Which if it be in lines, may be reduced into one Square, and from the side of that Square, take half the Coefficient, and the remainder shall be a. Which was the thing desired.

Gibson was influenced by Harriot, a fact which is indicated in the next chapter by an explicit citation, giving the resolution of AEquations, according to the generall Method composed by Mr. Tho. Harriot. Cajori was convinced that "Harriot had no general grasp of the theorem on the relations between roots and coefficients,"[219] and yet we find Gibson in the second chapter, making the statement that[220]

For all Quadratique AEquations of this kinde (meaning the 3 types mentioned above), ... have two Roots, which being both together equall to the Coefficient...

He then completes the sentence and demonstrates the third equation geometrically.

...if upon the Coefficient, as a Diameter, a Semicircle be described, and the side of ff (the quantity given) be applied therein, perpendicular to the Diameter b, the two segments of b are the two Roots sought.

For in the AEquation -aa + ba = ff, it is by the 14th. of the 6th. of Euclid, as followeth.

b - a f" f'a"

[219] Cajori, Revaluation, p. 322.
[220] Syntaxis Mathematica, 1655, p. 27.

Wherefore either Segment may be a, and the other will be b - a, and f a mean betwixt them.

Observe the appearance of the primes. Gibson explains that reduction of equations is accomplished by "adding all that is necessary or subtracting all that is unnecessary on both sides of the sign of equality, or by subtracting contradictories if they happen on one and the same side, until the equation, purged of all unnecessary members, remains with all that is absolutely known on one side, equal to all that is unknown on the other."

$$aa - ba + dc + ba = gg + ba - dc.$$[221]

is used as the example. In the first member a Want is expressed by -ba. In order not to change the value of the equation, ba must be added to the other side, resulting in

$$aa + dc + ba = gg + 2ba - dc$$

Now ba is subtracted from both sides, giving

$$aa + dc = gg + ba - dc$$

Once more he subtracts ba from both members, leaving

$$aa - ba + dc = gg - dc$$

Lastly, that the consonants, or known quantity may all come to one side, Gibson subtracts dc from both sides, leaving the final equation in the form of[222]

$$aa - ba = gg - 2dc$$

Take the Rectangle 2dc out of the Square gg, and let the rest be a Square, namely ff.

Then it is reduced,

$$aa - ba = ff$$

Gibson must have been an excellent instructor for he now explains that

Having gone a little about, only for exercise of them that are quite unskillful herein, now they shall see this Reduction might have been quickly done another way,...

by freeing the expression of -ba and ba because they are Contradictories. Then he subtracts the dc and the remaining ba from both sides, thus reducing the equation in two steps.

Although he gives Harriot's method, he explains it in more detail and, at the same time, reduces its length. For purposes of comparison, the one example he gives is repeated here.

$$aa = ff$$

or "let it be exhibited in numbers, aa = 69169, arriving at the root."[223]

So that a being a single letter cannot fitly represent that Root, but some trinomiall, as is b + c + d should be put equall to a, and the square thereof should be equall to aa, or 69169.

But because it may be done as well by adding the Gnomons, that is repetition of the second working... so often as the ponits are more than two; a binomiall will serve with lesse trouble to do the same.

Let that Binomiall be b + c

And put b + c = a

Their Squares shall be therefore equall
That is, bb + 2bc + cc = aa
That is, bb + 2bc + cc = 69169

The Resolution

		. . .
The Homogeneall number given		69169
First single Root b = 2 and bb =		4.0000
Which 4.000 being subtracted from		
the number given 69169, then there		
Remains of the Number given		29169
Root decuplate b = 20		
Divisor 2.b 40.00		
The second single root c = 6		
2.bc	240.00	
cc	36.00	
	276.00	
Subtract		276.00
Remains of the Number given		1569
The Root increased b = 26		
The Root increased and decuplate b = 260		
Divisor is 2b = 520		
The third single Root c = 3		
2 bc	1560	
cc	0009	
Total	1569	
Subtract		1569
Remains of the Number given		0000

[221] Syntaxis Mathematica, 1655, p. 30. [222] Ibid., p. 31. [223] Ibid., p. 32.

The Root increased 263, is therefore the true Root, as may be proved by recomposition or by multiplying 263 by 263,...

The explanation of the added ciphers in each case is followed by the reason of the work given above.[224]

The reason of it depends upon the 4th. Prop. of the second Book of Euclide, where it is demonstrated, that If a right line be divided by chance into two parts, the Square made of the whole, is equall to the Squares of the parts, and to the Rectangle made of the parts twice.

So it is here as followeth.

The Square of the greater part, that is, of 260	bb = 67600
The Square of the lesser part, that is, of 3	cc = 00009
The Rectangle of the parts, that is, 260 into 3 twice.	2bc = 01560
Equal to the whole square	69169

Moore calls this the method of The Extraction of the Square Root by the Bones.

Gibson devotes the remainder of the text to cubics, some quartics and geometrical applications.

RAHN (1659) devotes the first part of his Algebra to Algebraic Arithmetic and the second part to Resolution of Problemes. The Use of this Algebra. The first chapter deals with Equations and their Properties: and first, How many Roots any Equation may contain.[225]

As great as is the Number of the highest Power in any Equation, so many Roots doth it contain; either Affirmative or Negative, or Impossible. These Negative Roots Des Cartes cals False Roots ..., and in Geometrical Projection they require a situation contrary to the Affirmative Roots; therefore they are no less true than the Affirmative ones. For which cause I forbear to call them False Roots, and content myself with the Term of Negative Roots, according to the import of the Sign [-] prefixed to such roots. Roots that are Impossible, are such as by their own Nature or their Signs are so ordered as that they render the Probleme unreasonable and impossible. Such are marked with the Sign [∋I].

This sign in the German edition is expressed by ✱. He then gives a quartic equation which he states may be divided by x - 4, x - 3, x - 2, and x + 5 and therefore

$$x = \begin{Bmatrix} 4 \\ 3 \\ 2 \\ -5 \end{Bmatrix},$$

as may be easily tried.

The next chapter is entitled, Of casting away the Second Term of the Equation. After a brief explanation he proposes the equation[226]

$$zz - 6z - bb = 0.$$

Here the Power of the first Term is 2 and the known quantity of the second Term 6. This divided by 2, gives 3, which taken from the Root z, leaves z - 3 equal to any quantity taken at pleasure, as suppose that be x, then

z - 3 = x and		1 |
z = x + 3		2 | 1 + 3
zz = xx + 6x + 9	The new root 3	3 | 2 ⊙ 2
6z = 6x + 18	x is 3 less	4 | 2 * 6
zz - 6z = xx (*) - 9	than the root z	5 | 3 - 4
zz - 6z - bb = xx - 99 - bb = 0		6 | 5 - bb

This is followed by an explanation of the different methods of reduction, that is by addition, [z - a - 13 = 20; z - a = 33], by subtraction, [z + a + 13 = b; z + a = b - 13], by multiplication, [$\frac{z + a}{3}$ = 29; z + a = 87.] and so on by division, involution, and evolution. Then comes a chapter devoted to rationalization of problems contained in the Principia Math. Univers. by Schooten. Finally, the author gives solutions of "divers Arithmetical and Geometrical Problemes" where he uses small letters for unknown quantities and capitals for the known.[227] This he says is not Descartes' way.

Problem 1. There be six quantities [a + b : a - b : ab : $\frac{a}{b}$: aa + bb; and aa - bb :].

[224] Syntaxis Mathematica, 1655, p. 35.
[225] Rahn, Algebra Tr. by Brancker, 1668, p. 47.
[226] Ibid, p. 49.
[227] Ibid., p. 56.

Now by any two of these given, find the other four.

For { a + b, a - b, ab, a/b, aa + bb, aa - bb } substitute { D, E, F, G, T, R } a = ? b = ?

First, then, by D and E (that is, the Sum and Difference of two quantities) given, find F, G, T, R, (that is, the Product, Quotient, and Sum and Difference of the Squares).

a = ? b = ?	1 2	a + b = D, a - b = E as in the Probleme.
1 + 2	3	2a = D + E
3 + $\bar{2}$	4	$A = \frac{D + E}{2} = A$
1 - 2	5	2b = D - E
5 + $\bar{2}$	6	$B = \frac{D - E}{2} = B$
or 1 - 4	7	$B = \frac{D - E}{2} = B$
ab = ? 4 * 6	8	$AB = \frac{DD - EE}{4}$ = the Product = F
$\frac{a}{b} = ?$	9	$\frac{A}{B} = \frac{D + E}{D - E}$ = Quotient —— = G

And so A and B are both explained by D & E, which was first to be found. The rest follows easily.

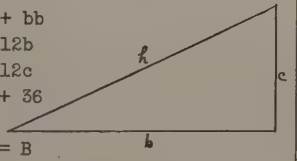
Etc.

An understanding of the method he uses in solving practical problems may be obtained by referring to the section on simultaneous equations.

Prob. V will illustrate this application to geometric problems.[228]

In an equicrural right angl. Triangle h + b = 6, how much is each?

h = ? b = ? c = ?	1 2 3	h + b = 6, b = c, bb + cc = hh (by the figure)
	4	h = 6 - b
4@ 2	5	hh = 36 - 12b + bb
3 - 5	6	o = cc - 36 + 12b
2, 6	7	o = cc - 36 + 12c
7 + $\overline{72}$	8	72 = cc + 12c + 36
8ω 2	9	$\pm\sqrt{72} = c + 6$
9 - $\bar{6}$	10	$\pm\sqrt{72} - 6 = c = B$
1 - 10	11	$12 \pm \sqrt{72} = H$



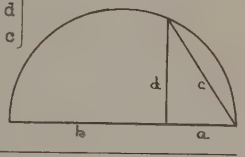
It must be remembered that @ means involution and a number in the outer margin with a vinculum above indicates a constant, and not a number of one of the items. The ω indicates that s root is to be taken.

Problem IV sets forth a trigonometric problem in

A Sine cutteth the diameter in extream and mean proportion, and the double sine of half the angle = 150 - 30$\sqrt{5}$. I would know the versed Sine, Cosine, and Radius.

Set for { Versed sine a, Cosine with Rad. b, Diameter a, Right Sine d, Double Sine ½ Angle c } b

a = ? b = ? c = ?	1 2 3	c = 150 - 30$\sqrt{5}$, a + b. b::b.a, c = b



He then proceeds as above.

Rahn was as familiar with the factor method of solving quadratic equations as Harriot, and arrives at the roots by equating each factor to zero. He recognizes a perfect square root and assigns $\pm$ to the result after extracting the square root. The double sign plus or minus, however, is placed a little above and below the square root. The square root sign is the vinculum added to the old sign $\surd$ to indicate its extent. Rahn also equates to zero. In the Dutch edition higher powers are obtained by multiplication; the additive principle is used by Brancker in the English translation of Rahn's text. Although neither of these editions has added anything new to this science, both serve as splendid indices to the rapidity with which development took place.

In De Vervulling van de Geometria en Algebra by ABRAHAM DE GRAAF, Amsterdam, 1708, which was examined in lieu of his earliest edition, the symbolism of Descartes is used. He adheres, however, to the Dutch method of indicating the second power of any expression by following it with the square root sign. This is illustrated on page 46.

[228] Rahn, Algebra Tr. by Brancker, 1668, p. 69.

z - 2 ☉ x	z - 2 = x
zz - 4z + 4 ☉ xx,	$z^2 - 4z + 4 = x^2$
- 2z + 4 ☉ -2x,	- 2z + 4 = -2x
-15 ☉ -15	-15 = -15
xx - 2x - 15 ☉ o	$x^2 - 2x - 15 = o$
Whence 5 en -3	5, -3 are roots
adding 2 2	
7 en -1, the values	

of z which satisfy the original equation.

De Graaf is not loath to use both positive and negative roots.

4. GEOMETRICAL DEMONSTRATIONS

That geometrical solutions of quadratic equations were familiar to the Greeks is now well established. "Hippocrates (fifth century), writing on the quadrature of the lines, in an attempt to square the circle, assumes a construction which is equivalent to the solution,"[229]

$$x^2 + \sqrt{\tfrac{3}{2}}\, ax = a^2.$$

Several propositions of Euclid present geometrical equivalents of the solution of various types of quadratic equations, not involving negative coefficients, and further study of similar problems appears in Euclid's Data.

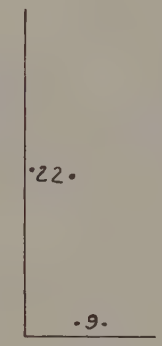
The geometrical solutions of quadratic equations presented by the Arabic writer Al-Khowarizmi more than a thousand years ago may be used with profit in the classroom today. Indeed, many of the authors whose works have been considered here included geometrical demonstrations in their texts. Among them were Perez de Moya, Bombelli, Brucaeus, Buteo, Digges, Unicorno, Coutereels, Petri, Menher, Lochner, Stampioen, Clavius, Ghetaldi, De Billy, Stevin, Vieta, Descartes, Oughtred, Renaldini, and Brasser. The demonstrations of some have already been cited. Two of Menher are given in Fig. 56.

LOCHNER makes application of the quadratic equation to the Pythagorean Theorem in his Tractalein (1583). On page 12 of this text the following problem may be found.

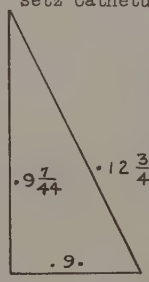
Das erste Theyl hat in sich exempla von zweyen Linien/die da machen ein rechtes Winckelmass/unnd wie man die lenger zwingen soll/dass sie auff die kürtzer dess einen orts orthogonaliter falle/und ist die Frag/ wie lang die Linien Catheti und Hypotenusae, ein sede insonderheit sense.

The first part contains examples of two perpendicular lines, and how the longer one should be swung on the shorter so as to form a right angle at one end, and the question is to find the length of the vertical line and the hypotenuse.

Das ander Exempel



Est die operation dem nechsten gleich/allein das die lenger Lein 22. Schuch/und die kürtzer 9. Schuch hat/ist auch die frag wie im nechsten/ folgt die operation/ich setz Cathetus sey 1. ℞ so ist Hypotenusa 22. ☉ - 1.℞. nun multiplicir ich 22. - 1.℞. inn sich selbss das ist quadrate also:



22☉ - 1.℞
22☉ - 1.℞
484.☉ - 22.℞.
- 22.℞. + 1.ʒ.
thut 484.☉ + 1.ʒ. - 44.℞.

Another Example

The operation is the same as the preceding except that the longer line is 22 and the shorter 9, and the question is as in the preceding. The operation follows: Set the vertical side equal to x so that the hypotenuse is 22 - x. Now multiply 22 - x by itself, that is, square it, thus:

22 - x
22 - x
484 - 22x
$- 22x + x^2$

giving $484 + x^2 - 44x$

and is the square of the hypotenuse. Square also the vertical side, x becomes x^2, then add to 81, that is the square of the base, becomes $81 - x^2 = 484 + x^2 - 44x$, solve and 403 = 44x, divide 403 by coefficient of x becomes $x = 9\frac{7}{44}$, and the vertical side has so many and the hypotenuse $12\frac{37}{44}$.

It is noteworthy that Lochner carefully checked this problem.

Und ist das Quadrat Hypotenusae, quadrir auch Cathetum, als 1.℞. wirdt 1.z. den addir zu 81. als zum Quadrat Basis, werden 81.☉ + 1.ʒ. gleich 484.☉ + 1.ʒ. - 44.℞. vergleichs mit einander/so

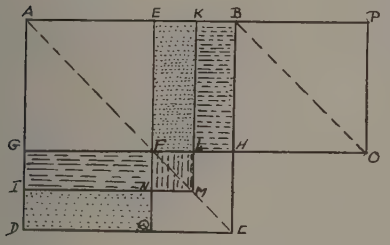
[229] L. C. Karpinski, Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi, 1915, p. 3.

wirdt 403.♃ gleich 44.℞ diuider durch ℞. wirt 1.℞. $9\frac{7}{44}$. und so viel Schuch soll haben Cathetus unnd Hypotenusa $12\frac{37}{44}$.

STAMPIOEN demonstrates the solution of 1 ② - 6 ① ghelyck -8, ($x^2 - 6x = -8$) as follows:

'T Bewijs [230]

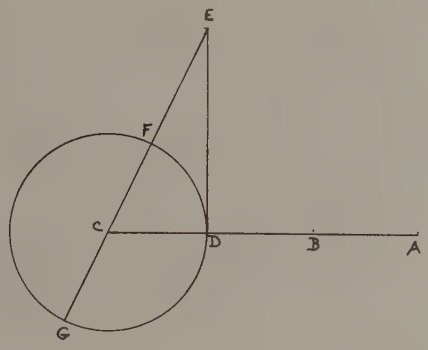
Om desen handel te beuysen, soo laet daer toe bereydt zijn het vier-cant ABCD. dat gesneden is met de linien EQ. ende GH. even-wydigh teghen te zynden. Dit soo zijnde, soo nemen wy dat de zyde des vier-cants AEFG als AE ofte EF doet 1①, soo doet sijn inhoudt 1 ②. daer by ghedaen een recht-hoeck, diens inhoudt is 4 ①, die wy stellen te zijn EBHF, daer van EB doet 4, (het ① ghetall) soo is EBHF 4①. Comt dan voor den gantschen recht-hoeck ABHG 1② + 4① die ghelych gheseydt wordt 60. Neemt nu de helft van EB, comt EK ofte KB 2, diens vier-cant is 4. voor FLMN, 't selfde ghedaen tot den winckel-haeck KLFNIA die noch ghelyck is 't recht-hoeck ABHG (om voorsaecke dat KH, daer af ghenomen is, ende gevoeght aen GF, wordende alsoo FI) comt het vier-cant AKMI 64. diens vier-cant-wortel is 8 voor AK. daer van ghetrocken EK 2. rest AE 6. de waerde des 1 ①.



Maer wanner het ① ghetall - by sich heelt, soo is AB de waerde des 1① ende BQ het ledighe. Ende wanner het ① ghetall met t'samen het ledighe - by sich hebben, dan so zijn FE ende EP ofte BA. de waerden des 1①, ende FP ofte ED het ledighe. Dat wy aldus voor-ghenomen hadden te bewysen.

Here ABCD is a square with two equal lines EQ and GH so drawn as to form the square AEFG, making AE or EF equal to x. AEFG is then equal to x^2. He makes the rectangle EBHF equal to 4x and since EF equals x, EB must be 4. The rectangle ABHG is then $x^2 + 4x$ which is equal, he says, to 60. Taking $\frac{1}{2}$ of EB, gives EK or KB equal to 2. The square of 2 is 4, or the square FLMN. The area KLFNIA equals the rectangle ABHG, from which the area KH has been subtracted and to which the area GN has been added. The square AKMI now equals 64, whose square root is 8 or the value of AK. Subtracting EK or 2 from it leaves AE or 6 as the value of x.

Demonstrations similar in aspect and equally as interesting are given by the other authors. One which results in the quadratic formula is selected from GHETALDI'S works because of its resemblance to one of Descartes'.[231]



First rule. $a^2 + ab = z^2$. So that one has, according to rule of algebra,

$$a = \frac{b}{2} \pm \sqrt{z^2 + \tfrac{1}{4}b^2}.$$

Now follows the geometrical demonstration

$$\overline{AB}^2 + AB.\ CB = \overline{DE}^2.$$

$$\overline{DE}^2 = \overline{EF}^2 + EF.FG.$$

This follows from the conditions of construction with D as mid-point of CB and ED perpendicular to CB.

$$\overline{DE}^2 = EF.\ EG = EF(EF + FG) = \overline{EF}^2 + EF\ .\ FG.$$

Since, however, $CB = 2CD = 2CF = FG$. It follows that $\overline{EF}^2 + EF\ .\ CB = \overline{DE}^2$. For the second rule $a^2 - ab = z^2$ where

$$a = \frac{b}{2} \pm \sqrt{\tfrac{1}{4}b^2 + z^2}$$

and for the third rule, $ab - a^2 = z^2$, where

$$a = \frac{b}{2} \pm \sqrt{\tfrac{1}{4}b^2 - z^2}.$$

[230] Algebra, 1639, p. 46.

[231] Abhandlungen zur Geschichte der Mathematischen Wissenschaften, 1882, Vol. IV, p. 207.

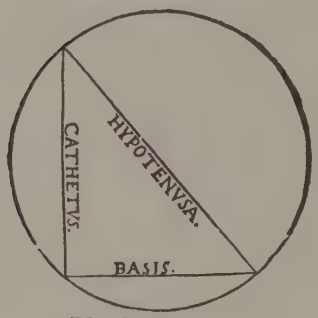
Tractätlein/

Darinnen etliche schöne

Exempel/ auß der Geometria/ zu dem Feld-
messen/vnd anderm sehr nützlich vnd dienstlich/ durch
die Edlen Regel Algebræ. die man sonst Coß nennet, mit
einem vortheil/ daß sie außer der Regel Al-
gebræ, oder Coß/ wol können ge-
macht werden:

Durch

Zachariam Lochner/Amatorem Musarum,
Burger zu Nürnberg/geordnet.



Nürnberg/ 1583.

Fig. 54. Title page of Lochner's Tractatlein (Nürnberg, 1583) together with two pages from the same text demonstrating geometric applications of algebra.

66 Das vierdte Theyl

PROBA.

Quadrir den lengern Cathetum, als 16. vnd sein basis als 12. wirdt 256. vnnd 144 addir die 2. Quadrat / wirdt 400 vnnd ist das Quadrat der Hypotenusa, quadrir auch den kürtzern Cathetum vnd sein basis, das ist 12. vnd 16. wirdt 144. vnd 256. addir die 2. Quadrat/thut 400. darauß Radix quadrata, thut 20. vnnd so lang sein die 2. Linj Hypotenusa, ein jede insonderheit / vnd ist die Frag auffgelöst vnd probirt.

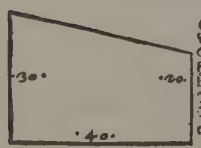
Vergehend Exempel

practice.

Multiplicir die 2. auffsteygenden Lini / sampt jrem basis quadrate, das ist 16 12. vnd 28. thun 256 144. vnnd 784. addir die grössern 2. Quadrat/als 784. vnd 256. thut 1040. subtrahir den kleinern Quadrat/als 144 bleibt 896. die 896 medir/wirt 448. die 448. theyl ab durch die zahl basis, das ist/ mit 28 kompt 16. Schuch/vnd ist das grösser theyl basis, so muß von noht wegen das kleine theyl 12. schuch sein, vnd ist eben wie in der andern Operation.

Proba. Die Prob mag nicht anderst/ denn wie vorhin geschehen. So bekompst du beyde vergleichung der Hypotenusæ.

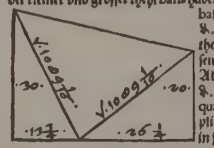
Das dritte Exempel.



Item/ich habe ein viereckte Figur/ sein die 2. auffsteygenden Lini ungleich/ hat die eine 30. die ander 20. vnnd basis 40. Schuch an der leng/ Nun will ich auß den obern 2. Winckeln herab 2. Linien ziehen/ eine so lang als die ander / vnnd will

diſes Tractetleins. 67

will machen 2. triangulos orthogonios. ist die Frag/ wie vil Schuch der kleiner vnd gröſſer theyl basis haben werde/ Seis/ der kleiner theyl basis sey 1. ℟. so ist der gröſſer 40. ℙ. — 1. ℟. quadrir den kleinern theyl basis, wirdt 1. ♎. quadrir auch sein cathetum, als 30. wird 900. ℙ. Addir die 2. Quadrat wirdt 900. ℙ. + 1. ♎. das behalt / dann es ist quadratum Hypotenusæ, multiplicir auch das gröſſer theyl basis in sich selbs/ wie folgt:

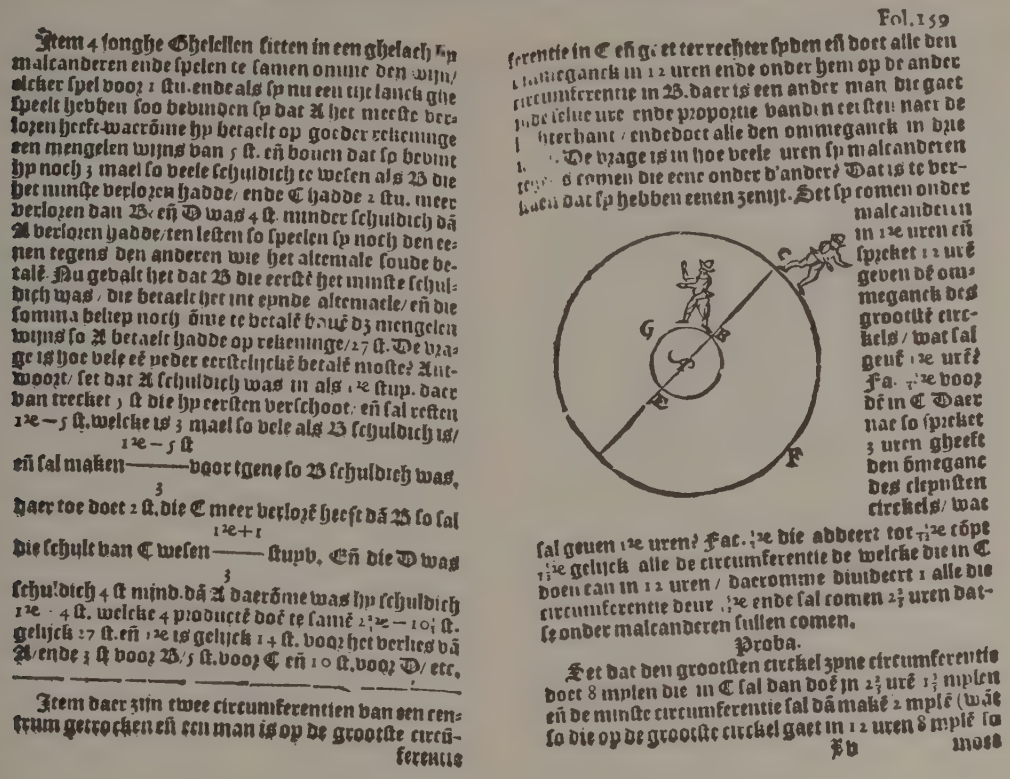


40. ℙ. — 1. ℟.
40. ℙ. — 1 ℟.
1600. ℙ. — 40. ℟.
— 40. ℟. + 1. ♎.
thut 1600. ℙ. + 1. ♎. — 80. ℟.

darzu addir das Quadrat seines Catheti, als 20. mal 20. thut 400. ℙ. ist zusamen 2000. ℙ. + 1. ♎. — 80. ℟. ist auch quadratum Hypotenusæ. dieweyl nun ein Hypotenusa der andern gleich muß sein / folgt das die Quadrat auch gleich an einander sein darumb wirdt gesprochen 2000. ℙ. + 1. ♎. — 80. ℟. ist gleich 900. ℙ. + 1. ♎. vergleichs wirdt 1100. ℙ. gleich 80. ℟. theyl ℙ. durch ℟ wirdt 1. ℟. 13¾. ℙ. vnnd ist das kleiner theyl basis, so muß das gröſſer not halben/ 26¼. Schuch sein.

PROBA.

Quadrir den lengern cathetum, als 30. vnnd sein basis, als 13¾. wirdt 900. vnd 189,¼ Addir die 2. Quadrat, wirdt 1089,¼. vnnd ist das Quadrat Hypotenusæ, quadrir auch den kürtzern cathetum vnd sein basis, das ist 20. vnnd: 6¼. wirdt 400 vnd 689,¼. Addir die 2. Quadrat/wirdt 1089,¼. da siheſt du, daß die 2. Quadrat Hypotenusæ, ein jeder in sonderheit dem andern gleich ist, vnd ist die Frag auffgelöst vnd probirt.



Fol.159

Item 4 songhe Ghelellen litten in een ghelach op malcanderen ende spelen te samen omme den wijn/ alcher spel voor 1 stu. ende als sp nu een tijt lanck ghe speelt hebben soo bevinden sp dat A het meeste verlozen heeft-warrôme hp betaelt op goeder reckeninge een mengelen wijns van 5 st. eñ bouen dat so bevint hp noch 3 mael so veele schuldich te wesen als B die het minste verlozen hadde/ ende C hadde 2 stu. meer verlozen dan B. eñ D was 4 st. minder schuldich dã A verlozen hadde/ten lesten so spelen sp noch den eenen tegens den anderen wie het altemale soude betalé. Nu gedalt het dat B die eerstê het minste schuldich was/ die betaelt het int eynde altemaele/ eñ die somma beliep noch ôme te betalé bouê dz mengelen wijns so A betaelt hadde op reckeninge/ 27 st. De vraage is hoe vele eé yeder eerstelijcks betalé moste? Antwoort/ set dat A schuldich was in als 1^{2e} stup. daer van trecket 5 st die hp eersten verschoot: eñ sal resten $1^{2e} - 5$ st. welcke is 3 maal so vele als B schuldich is/ $1^{2e} - 5$ st eñ sal maken ——— voor tgene so B schuldich was. 3

Daer toe doet 2 st. die C meer verlozê heeft dã B so sal $1^{2e}+1$ die schult van C wesen ——— stupv. Eñ die D was 3 schuldich 4 st minder. dã A daerôme was hp schuldich 1^{2e} - 4 st. welcke 4 productê doet te samé $2\frac{1}{2}{}^{2e} - 10\frac{1}{2}$ st. gelijck 27 st. eñ 1^{2e} is gelijck 14 st. voor het verlies dã A/ende 3 st voor B/ 5 st. voor C eñ 10 st. voor D/ etc.

Item daer zijn twee circumferentien van een centrum getrocken eñ een man is op de grootste circûferentie in C eñ gaet ter rechter syden eñ doet alle den [illegible]ganck in 12 uren ende onder hem op de ander circumferentie in B. daer is een ander man die gaet [illegible]de selue ure ende proportie van den eersten naer de [illegible]derhant / ende doet alle den ommeganck in dzelue [illegible]. De vraage is in hoe veele uren sp malcanderen [illegible]s comen die eene onder d'andere? Dat is te verstaen dat sp hebben eenen zenijt. Set sp comen onder malcanderen in 1^{2e} uren eñ spreket 12 urê geven dê ommeganck des grootste circkels/ wat sal geuê 1^{2e} urê? Fa. $\frac{1}{7}{}^{2e}$ voor dê in C. Daer nae so spreket 3 uren gheeft den ômeganc des cleynsten circkels/ wat sal geuen 1^{2e} uren? Fac. $\frac{1}{4}{}^{2e}$ die addeert tot $\frac{11}{28}{}^{2e}$ cõpt $\frac{11}{28}{}^{2e}$ gelijck alle de circumferentie de welcke die in C doen can in 12 uren / daeromme diuideert 1 alle die circumferentie deur $\frac{11}{28}{}^{2e}$ ende sal comen $2\frac{3}{11}$ uren datse onder malcanderen sullen comen.

Proba.

Set dat den grootsten circkel zyne circumferentie doet 8 mylen die in C sal dan doê in $2\frac{3}{11}$ urê $1\frac{3}{11}$ mylen eñ de minste circumferentie sal dã makê 2 mylê (wãe so die op de grootste circkel gaet in 12 uren 8 mylê so

Fv moet

Fig. 55. The circumference problem from Petri's Practique omte leeren Rekenen (1605).

5. TYPICAL PROBLEMS

Certain applications of algebra in the works of the period under discussion are of special interest because of the repeated use that was made of them by many of the authors, or the close resemblance that some bear to problems in modern texts. De la Roche, Rami, Vieta, Clavius, Henrion, and De Billy give any number of problems on continued proportion, as well as those that require the division of a number into parts which are in a given proportion. A quotation from CLAVIUS is used to illustrate this.[232]

Tres numeros continue proportionales in data proportione inuenire, quorum quadrati in vnam summam collecti faciant datum numerum.

Three numbers in a given continued proportion are such that the sum of their squares is equal to a given number.

Sint inueniendi tres numeri continuè sesquitertij, quorum quadrati faciant 4329. Ponantur quaesiti numeri 9ϕ. 12ϕ. 16ϕ in proportione sesquitertia. Eorum quadrati 81ϕ. 144ϕ. 256ϕ faciunt summam 481ϕ aequalem 4329. Diuisis ergo 4329 per 481. fiet 1ϕ 9. & 12ϕ. 3. Ergo primus numerus positus 9ϕ est 27. secundus 36, & tertius 48. quorum quadrati 729. 1296. 2304. faciunt summam 4329.

9x, 12x, 16x, are the numbers in continued proportion

$$481x^2 - 4329x = 3.$$

Then the numbers are

27, 36, 48.

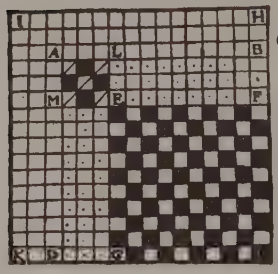
Another problem which is rather common is stated by Marco Aurel thus,[233]

[232] Arithmetica, 1609, p. 263.

[233] Ibid., 1552, p. III^{V}.

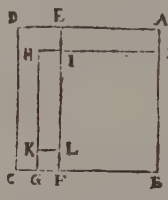
du nombre ☍ en ſoy, & en viendront 16, leſquelz adiouſtez auec 33, & feront 49. des meſmes tirez la √, & feront 7, aux meſmes adiouſtez 4, qui eſt la moitié du nombre ☍, & en viendront 11 pour la valeur d'vn ☍.

¶ *Demonſtration.*



Si vn ☍ eſt egal à 6 ☍ + 160, poſez q̃ A. L. ſoit la moitié du nombre ☍, q fait 3, leſq̃lz multipliez en ſoy, & en viendront 9, pour le quadrat A. L. M. E. aux meſmes adiouſtez 160, pource que le ſigne eſt d'augmentation, & il en viendra 169, deſquelz tirez la √ ☍, & en viendront 13 pour le coſté B. C. aux meſmes adiouſtez la moitié du nombre ☍, pource q̃ le ſigne eſt +, & en viendront 16, pour tout le coſté H. C. qui eſt la valeur d'vn ☍, & le ☍ vaudra 256, qui eſt pour tout le quadrat H. I. K. C. & pourtant que 6 ☍ + 160 ſont egaux à 256, tirez 160 de 256, & en reſteront 96, pour la valeur des 6 ☍, & 1 ☍ ſera 16, comme deſſus eſt dict.

¶ *Vne autre demonſtration.*



Si vn ☍ eſt egal à 6 ☍ + 160, faictes vn quadrat A. B. C. D. & tirez vne ligne E. F. paralelle auec A. B. que D. E. ou C. F. ſoit 6, & pour C. D. poſez 1 ☍. l'aire C. D. E. F. ſera 6 ☍, & l'aire A. B. F. E. ſera 160, qui ſont enſemble 6 ☍ + 160, egaux à 1 ☍, qui eſt l'aire de tout le quadrat A. B. C. D. prenez la moitié de D. E. 6, qui ſont 3, les meſmes multipliez en ſoy, & ſont 9, pour l'aire F. G. K. L. les meſmes adiouſtez auec B. F. E. A. 160. & feront 169, pour l'aire B. G. H. M. pource que l'aire H. I. K. L. eſt egal à l'aire A. E. I. M. des meſmes tirez √, & en viendront 13, pour B. G. l'vn coſté de ce quadrat, auec les meſmes adiouſtez G. C. 3. qui eſt la moitié de D. E. 6. & en viendront 16, pour la valeur d'vn ☍.

T 1 ☍

Fig. 56. Two geometrical solutions for the equation, $x^2 = 6x + 160$ from Menher's Practique Pour Brievement (1565).

Give a number which multiplied by $\frac{1}{2}$ with $\frac{1}{3}$ makes 54.

A problem similar to this is the division of a certain sum of money among several persons. We have already used this type of application to demonstrate the solution of either simultaneous or quadratic equations, and similar problems may be found in the texts of Gemma (1550), Marco Aurel (1552), Freyle (1556), Perez de Moya (1573), Buteo (1622), Unicorno (1598), Clavius (1608), De Billy (1637), Peletier (1554), Peucer (1556), Scheubel (1551).

The merchant problem was very popular at this time doubtlessly because of the vital interest in commercial activities. Such problems were given by Marco Aurel, Peletier, Unicorno, Clavius, Cataldi, and De Billy. As was mentioned before, cinnamon was the favorite commodity in problems of Italian authorship.

DE BILLY incorporated into his algebra the problem of the hare which was one of the typical ones included in early arithmetics. This LEYBOURN translates in the following manner.

Question II[234]

A Hare is 100 Geometrical paces distant from a Dog that swiftly pursues him, and the Dog runneth two times and a half faster than the Hare: It is demanded how many Geometrical paces the Hare will have run when the Dog overtaketh her; I put for these Geometrical paces 1 R; therefore the Dog which runs 100 paces more than the Hare, will have run 100 + 1 R: and for that the Dog runs twice and a half swifter than the Hare, I take two numbers in like proportion to one another; that is to say 5 and 2, and conclude that there is the same proportion between 100 + 1 R to 1 R, as between 5 and 2; therefore the product of the first number 100 + 1 R multiplied by the last number 2 (which is 200 + 2 R) is equal to the product of the two means

[234] Arithmetick, 1660, p. 350.

là deſſus qu'il auoit l'âge d'eux deux & quatre ans dauantage. Le Philoſophe Calliſthene ſe trouuant preſent à ce diſcours, vous me faites ſouuenir (fit-il) que mon Pere qui auoit 96 ans, auoit l'âge de vous trois. L'on demande icy quel âge auoit Alexandre lors qu'il eut ce pourparler, quel âge auoit Clite, & Epheſtion; Ie poſe pour les ans d'Epheſtion 1 R d'années d'où il s'enſuit qu'Alexandre en auoit 1 R † 2., donc Clite en auoit 2 R † 6 & ces trois enſemble ſelon la condition de la queſtion, doiuent étre égaux à 96, & partant il y a égalité entre 4 R † 8 qui eſt la ſomme des trois, & 96, ôtez en par tout 8 reſtera d'vn coté 4 R égales à 88, leſquels diuiſez par le nombre du plus grand caractere coſſique c'eſt à dire par 4 donnent le quotient 22 pour la valeur d'vne racine qui ſuppoſoit les années d'Epheſtion, donc Epheſtion étoit pour lors âgé de 22 ans, Alexandre de 24, & Clite de 50, qui tous enſemble font 96 ans.

QVESTION 2.

VN Lievre eſt diſtãt de 100 pas Geometriques d'vn Chien qui le pourſuit viuement, & le Chien court deux fois & demy plus viſte que le Lievre: L'on demande cõbien de pas Geometriques aura fait le Lievre lors que le Chien le ratteindra:

Ie poſe pour ces pas 1 R, donc le Chien qui aura fait 100 pas plus que le Lievre, aura fait 100 † 1 R, & par ce que le Chien court deux fois & demy plus viſte que le Lievre, je prens deux nombres qui gardent par entr'eux cette proportion, c'eſt à ſçauoir 5 & 2, & conclu qu'il faut qu'il y ait telle proportion de 100 † 1 R à 1 R, que de 5 à 2, donc le produit du premier nombre 100 † 1 R par le dernier 2, qui eſt 200 † 2 R eſt égal au produit des deux moyens 1 R & 5 qui ſera 5 R: donc en ôtant par tout 2 R reſteront 200 égaux à 3 R, & partant je diuiſe 200 par 3 qui eſt le nombre du plus grand caractere, & j'ay pour quotient $66\frac{2}{3}$ qui ſera la valeur de la racine. Ie dy donc que le Lievre aura fait 66 pas Geometriques, & $\frac{2}{3}$ quand le Chien le ratteindra, & le Chien en aura fait $166\frac{2}{3}$ qui font deux fois & demy dauantage que $66\frac{2}{3}$.

QVESTION 3.

L'Architecte Vitruue raconte en ſon Liure 9. Chap. 3. qu'Archimede trouua la quantité d'argent qu'vn Orfevre auoit mélé dans vne couronne d'or qu'il auoit faite au Roy Hieron, qui s'étoit engagé par vœu de la preſenter aux Dieux, peſante 100 liures. L'on demande comment Archimede eſt pû venir à bout de cela: l'opinion commune

Fig. 57. Typical problems from De Billy's Abrege des Precepts d'Algebre (1637).

1 R and 5 (which will be 5 R) therefore if you take from both parts 2 R, there will remain 200 equal to 3 R, and therefore I divide 200 by 3 which is the number of the greatest character and find in the quotient $66\frac{2}{3}$ which will be the value of the root. I say therefore that the Hare will have run $166\frac{2}{3}$ paces, which make twice and a half more than $66\frac{2}{3}$.

W. P. (HILIPS) in The Pathway of Knowledge, 1596, also gives a hare problem as follows:

A Gentleman hunteth a hare in the field, and he findeth by their tracing in the snow, that the hare had 60. lengths of the hounds passes before the houndes, and as often as the hare runneth 8 passes, the hound runneth but 6 passes, but two hounds passes are as much as 3 hares passes, the question is in how many passes the hound shall overtake the hare.

23 3 Hares paces.

If 6 hounds paces be —— 8 hares paces, what 2 houndes paces makes $2\frac{2}{3}$ hares paces.

If 3 hares paces be 2 hounds paces, what is $\frac{1}{3}$ hares paces make $\frac{2}{9}$ $\frac{1}{3}$ houndes paces.

Then say if $\frac{2}{9}$ paces be gotten by 2 houndes paces, how much shalbe gotten by 60 paces,

facit $12\rho - \frac{2}{9} - 2 - 60 - 12\rho$ facit $\frac{2}{9}2\rho$ like 120,

and 12ρ is like 540 paces of a hound, wherein he overtaketh the hare.

Otherwise.

Say if 2 houndes paces make 3 hares paces, what shall 60 houndes paces make, it maketh 90 hares paces, then say if 2 houndes paces by 3 hares paces what are 6 houndes paces, facit 9 hares paces, out of which take 8, there resteth one, then saye if 1 hares pace be gotten by 6 hounds paces, in how many houndes paces will the 90 hares paces be gotten facit 540 hounds paces.

The age problem, which may be called the "Alexander problem," proved very popular, having been incorporated into the works of Peletier (1554), Peucer (1556), Salignacus (1580), Rami (1599), Petri (1605), De Billy (1637), Follinus (1622), Lauremberg (1643), Ens (1651), and Leybourn (1659).

LEYBOURN gives it as the first problem on page 349 in the Algebra Brevis of his Arithmetick which is a translation of De Billy's text (see Fig. 5):

Alexander one day told Ephestion, that he was elder than him by two years; thereupon Clitus tells them, that he was as old as both of them (their ages added together) and four years over and above. The Philosopher Callisthenes being present at this discourse (saith he) I will remember that my father, who was 96 years old, had the age of you three. It is demanded here, how old Alexander was when he held this discourse, as also how old Clitus and Ephestion were. I put for the age of Ephestion 1 R of years, whence it follows that Alexander had 1 R + 2; therefore Clitus had 2 R + 6, and those three together, according to the condition of the question ought to be equal to 96: therefore there is an equality between 4 R + 8 (which is the sum of the three ages) and 96, take away 8 from both parts of the equation so there will remain on one side 4 R equal to 88, which divide by the number of the greatest Cossick Character, that is to say by 4, the quotient gives 22 for the value of one root, which was supposed for the age of Ephestion. Therefore Ephestion was at that time aged 22 years, Alexander, 24 and Clitus, 50, which altogether makes 96 years.

On page 351 of the same text Leybourn includes the problem of Archimedes.

Question III

The Architect Vitruvius in his ninth Book Chap. 3 tells us that Archimedes found the quantity of silver that a Goldsmith had mixed in a golden Crown which he had made for the King Hiero (who was obliged by vow to present it to the gods) weighing 100 pounds. It is demanded by what means Archimedes could arrive to the knowledge of that secret. The common opinion is that he took two masses, the one of gold, the other of silver, which weighed as much as the Crown; afterward he filled a vessel up to the brim with water, which vessel was placed in some grat (sic) bason, that the water that should be forced out of the first vessel, might be preserved and not lost. Thirdly, he gently put in the two masses and the Crown, one after the other, into the prepared vessel, taking notice of the quantity of water that issued out of the vessel at each time and concluding from thence that the Goldsmith had mingled 6 pounds and $\frac{2}{3}$ parts of silver. We will suppose then that the masse of gold weighing 100 pounds, did cast out of the vessel 60 pounds of water, & that the masse of silver also weighing 100 pounds, cast forth of the vessel 90 pounds of water, and that the Crown cast forth 65 pounds. I put forward for the silver mixed in the Crown 1 R, and constitute twice the Rule of three after this manner.

If 1 co pounds of gold give me 60 pounds of water, how much will 100 - 1 R? and I finde $\frac{6000 - 60\ R}{100}$ for my fourth number.

Secondly, if 100 pounds of silver give me 90 pounds, how much shall 1 R? and I find $\frac{90\ R}{100}$. Now these pounds of water $\frac{6000 - 60\ R}{100}$ and $\frac{90\ R}{100}$ added together do make $\frac{6000 + 30\ R}{100}$ pounds of water cast out, which ought to be equal to 65 pounds of water cast forth by the Crown, and therefore if we reduce them we shall finde 6000 + 20 R equal to 6500 (this reduction is made by multiplying the denominator 100 by 65 for seeing that this fraction $\frac{6000 + 30\ R}{100}$ is equal to 65 it shall be also equal to $\frac{65}{1}$, and therefore will be the same proportion of the numerator 6000 + 30 R to the denominator 100, as of the second numerator 65 to 1. (Marginally the author says, "Mark well this kinde of reduction once for all.") Therefore the product under the extreams 6000 + 30 R is equal to the product of the means 6500 take away therefore from both parts 6000, and there will remain an equation between 500 and 30 roots, and therefore divide 500 by 30, the number of the greatest character, you shall have the value of the root $16\frac{2}{3}$ for the pounds of silver mingled by the Goldsmith in the Crowns.

This same problem was found in the texts of Aurel (1552), Peletier (1554), Peucer (1556), Antich Rocha (1565), Clavius (1608), De Billy (1637), Lauremberg (1643), Ens (1651), and Coutereels (1658).

The work problem, which is so common in modern texts, was found in Menher (1565) and De Billy (1637).

The question of finding the dimensions of a wall was proposed by Gemma (1550), Clavius (1608) and Herigon (1634).

The army problem found may variations in the works of Scheubel (1550), Peletier (1554), Peucer (1556), Recorde (1557), Menher (1565), Visconti (1581), Digges (1594), Rami (1599), Clavius (1608),

Henrion (1615), Cataldi (1622), and Follinus (1622). The one used most frequently is number XLIII on page 330 of CLAVIUS' Algebra (1609). It reads,

In Caesar's army which is made up of Germans, Hungarians and Italians there are 35000 Germans. The Vngaris make $\frac{1}{2}$ of the Germans and Italians. The Italians make $\frac{1}{8}$ of the Germans and Hungarians. How many Hungarians and Italians are there and how many are there in the whole army?

Place $1\mathcal{Q}$ (x) for the number of Hungarians. Therefore the Germans and Italians are $2\mathcal{Q}$ and the whole army is $3\mathcal{Q}$. Since the Italians make $\frac{1}{8}$ of the Germans together with the Hungarians; but the Hungarians with the Germans makes $1\mathcal{Q}$ + 25000, then the Italians are $\frac{1}{8}$ + 3125 in number. Since, however, the Germans are 25000 and the Hungarians $1\mathcal{Q}$. subtracting $1\frac{1}{8}$ from each side, there will be an equation between 28125 and $1\frac{7}{8}\mathcal{Q}$. Dividing 28125 by $1\frac{7}{8}$, $1\mathcal{Q}$ will become equal to 15000, the number of Hungarians. The Italians are 5000 in number, namely $\frac{1}{8}$ of the Germans and Hungarians, which taken all together make 45000, the number in the whole army.

With no reflection on Clavius' ability as a mathematician, we wish, as an aside, to call attention to a faulty combination that Clavius made in Caesar's army. By the time the people on the Danube were recruited by Caesar, the Italians no longer manned his forces. When the Italians contributed to his army, the Hungarians were not a part of it.

Coutereels devotes many pages to various ways of solving the inheritance problem under the section which he entitles Reghel von Testamenten. He includes references on page 383 of his Arithmetica, 1658, to a similar problem in the texts of Tartaglia, Rudolff, Gemma Frisius, Forcadel, Ramus, Trenchant, Mellema, van der Gucht, Henrion, and van der Schuere. Later on he mentions Smyters, Peletier and Chauvet. Most of the works of these authors are arithmetics; but Coutereels himself solves problems both arithmetically and algebraically. Recorde (1557), Menher (1565), Unicorno (1598), Petri (1605), De Billy (1637), Lauremberg (1643), also include this problem in their algebras.

A question of a Testament by RECORDE (1557):[235]

A poore man died, whiche had fower children, and all his goodes came to .72. crownes: whiche he would have parted so, that the seconde and thirde childe should have .7. times as moche as the firste. And that the portions of the thirde and fourthe child should bee .5. tymes so much as the secondes parte: And that the first and the fourthe, should have twise as moche as the thirde. If you worke the solution wel, you maie seme worthy to be master of those wardes.

Scholar. I trust to obtaine moare benefite by the question, then by that office. Wherefore I will give good hede unto it. And for the first nōber, I set $.1.\mathcal{Q}$ then muste the seconde and thirde portions make together $.7.\mathcal{Z}\mathcal{Q}$. And the fourthe must bee all the reste of the .72. that is .72 - $8\mathcal{Z}\mathcal{Q}$. Now the thirde must be halfe the firste and fourthe, that is .36 - $3\frac{1}{2}\mathcal{Z}\mathcal{Q}$. And the third and fourthe, that is .36 - $3\frac{1}{2}\mathcal{Z}\mathcal{Q}$. And the third and fourthe, is .5. tymes the second. Wherefore the seconde shall be the .5. part of .108 - $11\frac{1}{2}$ that is .$21\frac{3}{5}$ - $2\frac{3}{10}\mathcal{Z}\mathcal{Q}$, whiche nomber I shall set in order with Letters, as here I have dooen for my owne ease, and aide of memory. And then shall I adde them all together. Whereof there commeth.

			1.
$129\frac{3}{5}$ - $12\frac{4}{5}\mathcal{Z}\mathcal{Q}$	A		
	B	$21\frac{3}{5}$.	$2\frac{3}{10}$
whiche is equalle to 72.	C	36.	$.3\frac{1}{8}$
First therefore I do adde all that foloweth - to bothe partes of the equatiō. And so have I	D	27.	.8.
		$129.\frac{3}{5}$	$12\frac{4}{5}$

$$129\tfrac{3}{5} = 12\tfrac{4}{5}\mathcal{Z}\mathcal{Q} + 72.$$

But bicause there are nombers Absolute on bothe sides, I shall abate the lesser somme, that is .72. from bothe partes, and then will there bee lefte,

$$57.\tfrac{3}{5} = 12\tfrac{4}{5}.1\mathcal{Q} \text{ that is.}$$

288. = .64.$\mathcal{Z}\mathcal{Q}$. And by diuision $4\frac{1}{2}$. = .1.$\mathcal{Z}\mathcal{Q}$. The Proofe.

So shall the firste mannes portion bee $4\frac{1}{2}$. And the seconde and thirde mannes portion .7. times so moche that is $.31\frac{1}{2}$. Whereby it followeth, that the fourthe manne, shall have the reste of 72. that is .36..

	A	$4\frac{1}{2}$	
	B	$11\frac{1}{4}$	$31\frac{1}{2}$
	C	$20\frac{1}{4}$	
Then seeyng the thirde manne, hath halfe so moche as the first	D	36	
		72	

[235] Whetstone of Witte, p. Hh. ij[r].

and the fourthe, his portion shall be $20\frac{1}{4}$. And then by diuerse reasons, the seconde manes part shall bee $.11\frac{1}{4}$. And all these partes added together, doe make rust 72. Wherefore the woorke is good.

After the word of commendation, "You have wroughte it well," Recorde solves the same problem in another manner. It might be well to note that the other authors remembered the wife in the last will and testament.

Petri and Menher give the wolf and lion problem. The latter illustrates many of the problems by beautiful wood cuts, depicting in one the peaceful relations that apparently existed between the two animals in question.

Some of the Dutch books had the circumference problem as shown in Fig. 55. The problem, as Petri (1605) states it, is

There are two circles which go around one center. The largest is called C on which a man goes to the right in 12 hours and the one under him goes around circle B to the left in three hours. In how many hours will they have the same zenith?

The answer given by the author is $2\frac{2}{5}$ hours.

THE SUMMARY

A comparative study of algebras covering a period of a little more than a century following the invention of printing has disclosed a quiet though persistent effort to standardize symbolism and methods in this field of mathematics. In addition to the presentation of a collection of illustrative material from practically every author listed in the bibliography, this study has brought to light various new facts bearing on the development of the science of algebra, which may be summarized as follows:

1. The pioneers in the development of algebra in this period were careful students of antiquity and were greatly influenced by the works of the ancient writers. This conclusion is based on the following facts:

a) There was exhibited a great dependence on Euclid, as is shown by the general use made of his geometrical demonstrations.

b) A great number of works on DIOPHANTUS was published together with frequent references to, or incorporation of, his problems in the texts of such outstanding authors as STEVIN and RENALDINI.

c) The Arabic influence, notably that of AL-KHOWARIZMI, either through LEONARD OF PISA, or through such translators as GHERARD OF CREMONA and ROBERT OF CHESTER, was very marked. This is attested to by the following statements:

1) Many authors employed LEONARD OF PISA'S classification of equations, although the Arabic types of $ax^2+bx=N, ax^2 = bx + N, ax^2 + N = bx$ were in general use as late as MOORE and DE BEAUNE.

2) The Arabic custom of designating unknown quantities by the squares of numbers was followed by some authors of this period.

3) The Arabic rule for the reduction of equations was the one generally employed throughout this period.

4) Whole sections of such texts as that of GHALIGAI were devoted to LEONARD OF PISA.

2. The writers on mathematics in this period were students of contemporary works. GRISIO was but one of the many writers who listed a large number of authors. Others, again, made reference in the prefaces to their textbooks, or in the text itself, to those who were most influential. Still others translated or adopted whole texts, as exemplified in the works of TAP and FOLLINUS. In this respect preference was given to PACIOLO, CARDAN, and STIFEL.

3. The symbolism employed for the unknown by the early writers was used, in general, until the introduction of the modern x, y, z by DESCARTES. However, the capital letters A, B, C, designating roots, were popular in the very beginning of the period under discussion.

4. Certain terminology employed today can be traced to this period. A term such as transposition, in the modern sense, was found as early as 1554 in the text of PELETIER. Other expressions as antithesis, depression, nugatory, long since obsolete, were popular in this period and continued to be used for some time after.

5. The modern method of solving radical equations was used with facility by NUNEZ, GOSSELIN, GHETALDI and others.

6. The modern exponential concept was introduced by CHUQUET. It was advanced by STEVIN and by GIRARD, whose symbols were employed by BOMBELLI and STAMPIOEN, and, as late as 1653, by BALAM.

7. Irrational roots were considered by BOMBELLI as early as 1579.

8. The double use of the radical sign, which received great support in the Netherlands, seems to have left no effect on present thought.

9. CLAVIUS and PETRI furnished the earliest examples of zero coefficients.

10. These same two authors seem to have been the first to consider negative roots and to use them freely.

11. PETRI can be credited with solving a quadratic equation by completing the square.

12. CLAVIUS and PETRI incorporated

in their texts the first examples in which quadratic equations are solved by factoring. This necessitated equating the quadratic member to zero.

13. The trial-and-error method of finding the roots of a quadratic equation was employed in this period by STEVIN, GIRARD, and COIGNET.

14. HARRIOT generalized the solution of the quadratic equation.

15. The modern quadratic formula owes its adoption to DESCARTES.

16. DESCARTES, in addition to all his other contributions, discovered the relationship between the equation and the geometrical locus whose coordinates satisfy the equation.

17. In the text of STAMPIOEN we find the first illustration in which the roots, positive and negative, were designated by the modern + and −.

18. This same author appears to have been the first to develop the present method of checking results by the direct substitution of the roots in the original equation.

19. VIETA was the most potent advocate of the classification of quadratic equations into pure and affected, although earlier authors, as HERIGON, used the term affected quadratics.

20. VIETA was one of the first to reduce affected quadratics to pure quadratics.

21. Literal symbolism received its impetus in the last thirty years of this period.

22. Authors of the last thirty years of this period left us a heritage in the discovery of the relationship that exists between the coefficient of the second term with its sign changed and the sum of the roots, together with that which exists between the third term and the product of the roots.

23. Eight out of the twenty-two items listed here were contributed by Dutch mathematicians.

A consideration of these facts seems to justify the conclusion that the period 1550-1660 presented the mathematical world with the systematic treatment of simple and quadratic equations which are incorporated in modern textbooks.

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4 p.ℓ., 303, 14, 201, 7 p. 15 cm. (UM).
Bound with Thavmatvrgi Physici Prodromvs, ad Cl. v. AEgidivm Gelenivm, Coloniae, 1649.

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1. Academia Algebrae darinnen die miraculosischen Inventiones zu den hochsten Cossen weiteres continuirt und proficirt werden. Ulm, 1631.
2. Numerus figuratus sive arithmetica analytica arte mirabili inaudita nova constans. ULM, 1616. (NYP).
3. Ingenieurschul, 2 Aufl., Nurnberg, 1633-1637.
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1. Scala Grimaldelli libro di aritmetica, e geometria Speculatiua, e Pratticale Di M. FRANCESCO FELICIANO Veronese, 1669.
   1 p.ℓ., [6], 240 p. illus. 21 cm. (UM).
   Ed. pr. 1526; Verona, 1602 (UM).
   In Rara arithmetica, SMITH tells us that this book appeared one hundred and forty-three years after it was first published. The University of Michigan has a copy of the edition, Venetia, 1692, which shows that this text

enjoyed longer life than expressed above. Professor KARPINSKI mentions FELICIANO in his article, "Arithmetic Centenarians: Textbooks with a long life." Scripta Mathematica, Vol. II, November, 1933, p. 34.

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DE MORGAN tells us that the first volume contains a treatise on Arithmetic, and on Cossic Arithmetic, or Algebra.

XXXIII. HERMANNUS FOLLINUS (1624-1685).
Algebra sive Liber de Rebus Occultis, quem praecedit Arithmetica Synthetica, sine quâ posterior nequit intelligi: Cum Duabus Orationibus quarum una est de Natura Febris Lenticularis sine de Petechiis: altera de studijs Hippocraticis cum Chynuciis coniungendis, utraque; habita in Avditorio Medico: prior quando publicas Medicinae lectiones auspicabatur Hermannus Follinus, Artium et Medicinae Doctor, olim Sylueducensium Mediciis Ordinarius, nunc in Alma Universitate Coloniensium Professor Publicus. Inuidia Comes Virtutis Coloniae, excudebat Gerhardus Creuenbruch. Anno M.DC. XXII.
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4 p.ℓ., 65 p. illus. 23 1/2 cm.
The first summary of arithmetic printed in America translated into English by D. E. SMITH. This has a series of algebraical problems.

XXXVII. GEMMA REINER, or REGNIER, FRISIUS (1508-1555).
Arithmeticae practicae Methodus facilis per Gemmam Frisium, Medicum ac Mathematicum, iam recens ab ipso authore emendata & multis in locis insigniter aucta...Paris, 1549.
96 numb. ℓ. incl. tables, diagrs. 17 cm. (UM).

With this are bound Köbel, Jacob. Astrolabii declaratio, 1550; Cosmographiae introductio, 1550.
Ed. pr. 1540 (NYP). The editions Witebergae, 1550, (UM) and Paris, 1550 (UM) are substantially the same. SMITH'S Rara cites 59 or more editions in the 16th century, besides the numerous editions after 1600. Editions, Venice, 1567 and 1576, are in the University of Michigan.
This is another of the authors included among the "Centenarians" by Professor Karpinski, op. cit. BIERANS DE HAAN mentions 19 distinct works by GEMMA.

XXXVIII. JOHN GEYSIUS.
The algebra proper in the Encyclopedia by John Alsted was written by JOHN GEYSIUS. (See Number I.)

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Practica d' arithmetica di Francesco Ghaligai Florentino. Nuouamente Riuista, & con somma Deligenza Ristampata, Florence, 1552.
1 p.ℓ., 114 numb.ℓ. 21 x 16 cm. (UM).
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1. De resolutione et compositione mathematica libri quinque. Opus posthumum, Rome, 1630 (UM).
2. Patritii Ragusini variorum problematum collectio, Venetiis, apud Vincent Fiorinam, 1607.

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Syntaxis Mathematica, London, 1655.
7 p.ℓ., (260) p. diagrs. 9 1/2 x 15 cm. (UM).

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1. Invention nouvelle en l'Algebre. Reimpression par DR. D. BIERENS DE HAAN.
4 p.ℓ., [64]p. diagrs. 21 1/2 cm. (UM).
This is a reprint of the original ed., Amsterdam, 1629. GIRARD also revised a work of SAMUEL MAROLOIS in 1628 (UM).
2. Les oeuvres mathematiques de SIMON STEVIN, revues, corrigés et augmentés, 1 vol. fol., Leyden, 1634.(UM).

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Decas prima exercitationum Mathematicarum, Neapoli, 1627.
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XLIV. GUILLAUME GOSSELIN (?-1590).
1. De Arte Magna, seu de occulta parte numerorum quae et Algebra et Amucabala vulgo dicitur, Paris, 1577.
This was taken from 'Le "De Arte Magna" de GUILLAUME GOSSELIN' by H. BOSMANS in Bibliotheca Mathematica, Vol. 7, p. 44. (UM).

2. L' Arithmetique de Nicolas Tartaglia Brescian, Grand Mathematicien et Prince des Praticiens...receuillie et traduite d' Italien en Francois. Paris, 1578.

XLV. ABRAHAM DE GRAAF (fl. 1650).

1. Vier boecken, drie vande drie hoeksmetinge en een vande telkunstigen. Amsterdam, P. Goos, 1659. (UM).
2. De beginselen van de algebra of stelkonst volgens de manier van Ren. Des Cartes. Amsterdam, 1672 (LMSA).
3. De geheele mathesis of wiskonst herstelt in zijn naturlikle gedaante: Amsterdam, J. ten Hoorn, 1694.
   8 p.ℓ., 318, [2] p. plates (part. fold.) diagrs. 25 cm. (UM). The third edition was published in Amsterdam, 1717, (UM).
4. Inleyding tot de wiskunst; of De beginselen van de geometria en algebra, door Abraham de Graaf. De 2. druk. Amsterdam, J. Loots, 1706.
   3 p.ℓ., 375 p. diagrs. 20 3/4 x 16 1/2 cm. (UM).
5. Analysis, of Stelkunstige ontknoping in de meetkungstige werstukken; vindende van hen de grootste en kleenste; de raaklynen op de kromme; de plaatsen; de ontbinding der bepaalde werkstuffen door de plaatsen; en de quadratura van eenige kromlinische grootheden. Amsterdam, J. Loots, 1706.
   4 p.ℓ., 391 p. incl. diagrs. 20 1/2 x 17 cm. (UM).
6. De vervulling van de geometria en algebra, begrepen in het boek genaamt de Inleyding tot de wiskunst: handelende van de voornaamste eygenschappen der kegelsneden en de ontbinding der aequatien van drie, vier en meer dimensien tot in't oneyndig. Amsterdam, J. Loots, 1708.
   2 p.ℓ., 144 p. diagrs. 20 x 16 cm. (UM).

XLVI. JOHANN RUDOLF VON GRAFFENRIED (1584-1648).

Arithmeticae Logisticae Popularis Libri IIII...In welchen der Algorithmus in gantzen zahlen und Fracturen sampt der Proportion, Neben der Welschen Practic alle andere dienstliche Regeln biss zu der Coss begriffen seynd und gantz verständlich mit schönen Fragen unnd Exemplen auff das einfaltigest erklärt. Also dass ein jeder so eines zimlichen Verstands diese kunst selber hierauss erlernen kan. Allen freyen Mathematischen kunstbegierigen und jedermenniglich zu gutem auss den fürnembsten Authoren zusammen getragen und zum ersten mal an Tag gegeben. Durch Johan Rudolff von Graffenried, HB. Freyer Mathematischen Künsten Liebhaber. Getruckt zu Bern bey Abraham Weerlj: in verlegung dess Authoris. 1619.
1 p.ℓ., [27], 704, [8]p. diagrs., tables. 20 1/2 cm. (UM).

XLVII. M. HENRICUS GRAMMATEUS (HEINRICH SCHREIBER) (circ. 1496).

Ein new künstlich behend und gewiss Rechenbüchlin uff alle Kauffmanschaft nach...Etlichen Regeln Cosse. 1535. (Col. U).

XLVIII. SALVATOR GRISIO.

Antanalisi a qvesiti Stampati nell' Analisi di Benedetto Maghetti. Opera Algebraica di Salvator Grisio della cava. In Roma, Apresso Francesco Caualli, 1644.
5 p.ℓ., 144 p. 22 1/2 cm. (UM).

XLIX. THOMAS HARRIOT (1560-1621).

Artis Analyticae Praxis, ad aequationes nova, expedita et generali methodo, resolvendas: Tractus E posthumis Thoniae Harrioti Philosophiae Mathematici celeberrimi Schediasmatis summaside & diligentia descriptus: Et Illustrissimo Domino Dom. Henrico Percio Northumbriae Comiti, Sondini, Apud Robertum Barker, Typographum Regium: Et Haered. Io. Billii. -1 vol. London, 1631.
3 p.ℓ., 180 p. 1ℓ. 23 1/2 x 34 cm. (UM).

L. GEORGE PHILIP HARSDÖRFER.
Delitae mathematicae et physicae. Nürnberg, 1651.

LI. DENIS HENRION (-1640(?)).
1. L'usage du compas de proportion. Paris, 1631.
13 p.ℓ., 290, 10 p. fold. p. diagrs. 19 1/2 cm. (UM).
2. Traicte D'Algebre. Paris, 1620. (NYP).
3. Les Quinze Livres des Elements D'Euclide. Traduicts de Latin en Francois: Avec un Sommaire & Abbrege de l'Algbre, qui sert à faciliter l'intelligence du dixiesme Livre. Par D. Herion Mathemat. A Paris, Chèz Abraham Pacard, rue sainct Jacques, à l'esto elle d'or. 1615. (UM).

LII. PIERRE HERIGONE (fl. 1640).
Cursus Mathematicus nova brevi et clara methodo demonstratus. Paris, 1634.
6 v. diagr. 18 cm. 2d. ed., 1644 (UM).
The supplement to volume six is entitled, Supplementvm Algebrae, 1644. The Supplement is bound in a separate volume, 1642. (UM).

LIII. GIELIS VAN DER HOECKE.
Een sonderlinghe boeck dye edel conste Arithmetica met veel schoone perfecte regulen als Die numeracie van den ghetallen metten specie int gheheele en ghebroken. ?, 1537.
4 p.ℓ., 176 numb.ℓ, tables, diagrs. 15 1/2 cm. (GAP).

LIV. HEINRICH HORCHEN.
Aufangs grunde einer Vernunstt - und Schrifftübenden Zahl - und Buchstab - Rechenkunst, deren diese sonst Algebra heisset...Leipzig, 1659.

LV. JOHN HUDDE (1633-1704).
John Huddeni Epist. I, de reductione aequationum. Amsterdam, 1659 (UM).
This is bound in with Geometria a Renato Des Cartes, Amsterdam, 1683. It deals chiefly with cubic equations and those of higher order.

LVI. FRANS VAN DER HUIPS.
Algebra; Ofte Een Noodige, korte en klare onder-wijsinge begin zelen en gronden vande Stel-Konst: Door-gaans verçiert met verscheiden nieuwe Regelen. Met een Aan-Haugh. Vervattende: 't eerste Stel-konstigh Boeck Diophanti Alexandrini. Nu eerst in onse Nederduitze Taal, vertaalt; konstigh verklaart ende ontbonden op verschieden manieren: door Stel-konstige getallen, ende nieuw door Vertogen in Tel-konstige gestallen. Voor de tweede-maal; met een Toe-voegingh, op d'ont-bindinge der Vierkants-vierkante Stel-konstige Vergelijkinge: ende door-gaans verbetert. Door Frans van der Huips. t' Amsterdam, Voor Hendrick Tjercksz. de Vries, Boeck-verkoper op de Zeedijck, over de Stormsteegh, in de Chronijck van Vriestandt, 1661.
6 p.ℓ., 181, [2], 70 p. diagrs. 16 cm. (UM).
There is an ed., 1654, in the Bibliotheque Royale, Bruxelles and Library of the Mathematical Society at Amsterdam.

LVII. JAMES HUME (fl. 1639).
1. Le traite d' Algebre. Paris, 1635.
2. Algebre de VIETE, D'une Methode Novvelle, Claire, et Facile, Paris, 1636.
8 p.ℓ., 552 p. diagr. 17 cm. (UM).
Columbia University has a copy of this work dated the same. However, it must be an earlier edition since the copy at the University of Michigan includes one more Theorem, namely Prop. 11, Theorem 9 on p. 550.

LVIII. GERARD KINCKHUYSEN.

1. De Grondt der Meet - Konst. Te Haerlem, 1660. Another edition, 1684. (UM).
2. Algebra ofte Stel - Konst beschieven tot dienst van de Leerlingen. Haerlem, 1661. (UM).
3. Geometria: Ofte, Meet - konst. Haerlem, 1663.
   165, [1]p. diagrs. 20 cm. (UM).

LIX. WIL KLEBITIUS.

Insvlae Melitensis, qvam alias Maltam vocant, Historia, quaestionib, aliquot Mathematicis reddita incundior. Antwerp, 1565.
Very rare.

LX. JOANNES LANTZ.

Institutionem Arithmeticarum libri quatuor...cum fractionum astronomicarum, etc. Coloniae Agrippinae, 1630. (BM).
The University of Michigan Library has photostats of this edition.
Other editions are 1616 and 1619.

LXI. PIERRE LA RAMÉE (RAMUS, RAMI) (1515-1572).

1. Arithmeticae libro duo: geometriae septem et viginti. A Lazaro Schoner recogniti & autem aubrium, 1599.
   8 p.ℓ., 240, 178,[4]p. 1ℓ. illus. folded tables, diagrs. 24 1/2 cm. (UM).
   Other editions of 16th century: 1569 (GAP), 1577 (NYP), 1580 (GAP), 1581 (NYP), 1586 (GAP), 1591, 1592 (Col. U), 1599, Frankfurt, 1627 (NYP).
2. Arithmetices et algebrae libri II a L. Schonero emend. Frankfurt, 1586 (GAP).
3. P. Rami Arithmeticae libro duo cum commentariis WILEBRORDI SNELLI R. F. Leyden, 1613.
   2 p.ℓ., 106 p. 19 1/2 cm. (UM).
   There is also an Arithmeticae libri tres, 1st. ed., 1555 (GAP).

LXII. JOHANN WILHELM LAUREMBERG (1590-1658).

1. Arithmetica, Peculiaribus observationibus & Exemplis Historicis illustrata. Itidem Algebrae principia. In Regia Academia Sorana proposita à Joanne Laurembergio, Mathem. Professore. Sorae, Exeudii Henricus Crusius, Acad. Typogr. 1643.
   4 p.ℓ., 223, [1]p., 2ℓ. diagrs. 19 cm. (UM).
2. Pet. Lavrembergi Rostochiensis Institutiones arithmeticae. Jenae, 1659.
   12 p.ℓ., 226 [2]p. diagrs. 13 cm. (UM).
   POGGENDORFF mentions an ed., Hamburg, 1621.
3. Disp. mathematica, Hamburg, 1623.

LXIII. WILLIAM LEYBOURN (?-1690).

Arithmetik, vulgar, decimal, instrumental algebraical. In four parts. London, 1660.
7 pℓ., 392 p. double tables, diagrs. 17 cm. (UM).
The fourth part: being an abridgment of the precepts of algebra. Written in French by JAMES DE BILLY: and now translated into English. With diverse questions added, which were not in the original. London, 1659.
The University of Michigan also has the edition,"carefully cor.; & very much enlarged by the author." London, 1700.

LXIV. ZACHARIAS LOCHNER.

Tractatlein, Darinnen etliche schoene Exempel ausz der Geometria, zu dem Feldmessen, vnd andern sehr nuetzlich vnd dienstlich durch die Edlen Regel Algebrae, die man sonst Coss nennet, mit einem vortheil dasz sie ausser der Regel Algebrae, oder Coss, wol koennen gemacht werden. Nuernberg, 1683.
4 p. ℓ., 92, (4) p. (UM).

LXV. RAPHAEL MAFFEI.
De Scientis mathematicis, 1511.

LXVI. SAMUEL MAROLOIS.
1. Meet - conste Verbetert door A. Girard: door E. de D. Amsterdam, 1629.
2. Opera Mathematica of Wiskundige Wercken door A. Girard. Amsterdam, 1628 (UM).
Other editions, 1b. 1630, 1633 (UM), 1638, 1662.

LXVII. THOMAS MASTERSON (16th century).
Arithmetic. London, 1592, 1634.
3 p.ℓ., 21, [4], 141, [9], 68 p., diagrs. 18 cm. Ed. pr. 1592-95, 1634 (GAP).

LXVIII. PETER MENGOLI (1625-1686).
1. Via Regia ad Mathematicas scientias per Arithmeticam, Algebram, & Planimetriam ornata. Bononise, 1655. (BM).
Photostats of this edition can be found at the University of Michigan.
2. Geometria speciosa, 1665 (UM).
Both of these books may be found in the Universitats - Bibliothek, Göttingen.

LXIX. VALENTIN MENHER DE KEMPTEN.
1. Arithmetique seconde. Antwerp, 1556.
2. Practique povr Brievement apprendre à Ciffrer, & tenir Liure de Comptes, auec la Regle de Coss, & Geometrie. Anvers, 1565. Ed. pr., 1550.
4 p.ℓ., [162]p., 11, [1] 6, 1, [276]ℓ. illus., diagrs. 17 x 10 cm. (GAP).

LXX. JONAS MOORE (1617-1679).
1. Arithmetic, vulgar and algebraical. London, 1660.
7 p.ℓ., 455 p. front. diagrs. on 7 p., (partly folded), tables. 17 cm. (UM).
Other editions are: 1650, 1668 (UM).
2. New Systeme of the Mathematicks. London, 1681. 2 v. (UM).
Bound with this is MOORE'S Arithmetic in Species or Algebra, 1680.
3. Mathematical Compendium. London, 1674.
The 2d. ed. published in 1681 is in the University of Michigan Library.

LXXI. PETER NUÑEZ (NONNIUS, NONNES) (1502-1578).
1. Petri Nonii Salaciensis Opera. Basel, 1592.
6 p.ℓ., 439, [1]p. illus. diagrs. 30 cm. (UM).
2. Libro De Algebra en Arithmetica y Geometria. Compuesto por el Doctor Pedro Nuñez, Cosmographo Mayor del Rey de Portugal, y Cathedratico Iubilado en la Cathedra de Mathematicas en la Universidad de Coymbra. En Anvers. En casa de la Buida y herederos de Iuan Stelsio. 1567.
16 p.ℓ., 341, i.e., [339] numbered ℓ., diagrs. 16 1/2 cm. (UM).
There was an ed., 1564, and two editions at Antwerp in 1567.

LXXII. JUAN DE ORTEGA (fl. 1567).
1. Tractado subtilissimo de arismetica y de geometria. Compuesto por el reuerendo padre fray Juan de Hortega de la orden de los predicadores. Ahora de nueuo emendado con mucha diligecia Honcalo Busto d' muchos errores que auia en algunas impressiones passadas. 1537 (UM), 1552 (HSA), 1563 (HSA).
2. Svma de Arithmetica; Geometria Practica vtilissima: ordinata per Johane de Ortega Spagnola Palentino. Ed. pr. Barcelona and Lyons, 1512; Lyons, 1515; Rome, 1515; Messina, 1522; 1534; Seville, 1536; 1537; Paris, 1540 (?); Seville, 1542; 1552; S.ℓ. (?), 1552; Granada, 1563.

GONSALO DE BUSTO published a new edition of the Arithmetic of ORTEGA (see Number XVI).

3. Opera, 1628 (UM) and 1638 (UM).

LXXIII. WILLIAM OUGHTRED (1574-1660).

1. Arithmeticae in numeris et speciebus institutio: Quae tum Logisticae, tum Analyticae, atque adeo totius Mathematicae quasi clavis est. Ad Nobillessimum Speclatissimumque invenem Dn. Guilalmum Howard, Ordinis, qui dicitur, Balnei Equitem, honorissimi Dn. Thomae, Comitis Arundelle & Surriae, Comitis Mareschalli Angliae, etc., filium Londini, Apud Thomam Harperum. 1631.
   1 p.ℓ., 6, 88 p. incl. tables, diagrs. 16 cm. (UM).
   The same with Tract de resolutione aequationum in numeris was published in London, 1648.
2. Clavis Mathematicae denuo limata, sive potius fabricata. Oxoniae, 1522.
   7 p.ℓ., 151 p., 1ℓ., 46 p., 2ℓ., 10 p., 1ℓ., 41 p. incl. plates, 5 p.ℓ. diagrs. 16 1/2 cm. (UM), ed., 1667 (UM). (Later Latin editions: 1648, 1652, 1667, 1693; English editions: 1647, 1694.)
3. The Key of Mathematicks new forged and Filed: Together with a Treatise of the Resolution of all kinds of affected aequations in Numbers...With the Rule of Compound Usury; And demonstration of the Rule of False Position. And a most easie Art of delineating all manner of Plaine Sun-Dyalls. Geometrically taught by Will. Oughtred. London, 1647.
   2 p.ℓ., [8], 174, 29 p. diagrs. (part fold.) 14 cm. (UM).
4. Opuscula mathematica inedita. Oxonii, 1677.
   4 p.ℓ., 212 p., 1ℓ. fold. tables, folded diagrs. 18 cm. (UM).
5. Mathematical Recreations. London, 1643.
   6 p.ℓ., [24], 286, [1]p. illus. diagrs. 16 1/2 cm. (UM).

LXXIV. LUCA PACIOLO (PACCIOLA) (1445(?)-1509(?)).

1. Summa de arithmetica, geometria, proporzioni e proporzioalita. Venice, 1494.
   224ℓ. illus. 32 cm. t. -p lacking. (UM).
   2d. ed., 1523, is a posthumous publication and differs from the first one in the spelling of first references to folio. (UM).

LXXV. JACQUES PELETIER (1517-1582).

1. L'Algebre De Iaques Peletier du mans, departie an deus Liures. A Tresillustre Segneur Charles de Cosse Marechal de France. Alion Par Ian de Tournes. 1554.
   [20], 229, [8]p. diagrs. tables. 17 1/2 cm. (UM).
   Editions, 1560, 1620, are in GEORGE A. PLIMPTON library.
2. De occulta parti numerorum. Paris, 1560.
   4 p.ℓ., 57 (i.e., 62) numb.ℓ., 11 p., 1ℓ. incl. tables, diagrs. 21 cm. (UM).

LXXVI. JUAN PEREZ DE MOYA (16th Century).

1. Trattato de Mathematicas en qve se contienen cosas de arithmetica, geometria, cosmographia, y philosophia natural. 1573.
   3 v. in 2. tables, diagrs. 27 cm. (v. 2-3, 28 1/2 cm.) (UM).
2. Arithmetica practica, y especulativa del bachiller Juan Perez de Moya, ahora nuevamente corr., y anadidas por el mismo autor muchas cosas. En Barcelona, En la impr. de R. Figuero, 1705.
   8 p.ℓ., 380 p. diagrs. 20 1/2 cm. (UM).

LXXVII. NICOLAUS PETRI, DAVENTRIENSIS, (fl. 1567-1588).

1. Arithmetica. Practique omme cortelijcken te leren cypheren, nae allerleye coophandelinghe, op Amstelredamsehe maete, enz. Amstelredamsehe maete, enz. Amsterdam, 1567.

2. Practique om te leeren rekenen, cypheren en de boeckhouwen met die regel coss ende geometrie. Amsterdam, 1605.
   7 p.ℓ., 289, [1], (10), [1] numb.ℓ., [1]p. 16 1/2 cm. (UM). (13), (3), (10) numb.ℓ., t.-p. in manuscript. Ed., 1635 (UM).
   Ed. pr., 1583 (LMSA); 2d. ed., 1591, 1596, 1603.
3. Geometria, en andere Questien van de Algebra Zijn vinder achten J. R. BRASSER, Regula Cos of Algebra. Amsterdam, 1663 en 1672. (UM).

LXXVIII. KASPAR PEUCER (1525-1602).
1. Valentini Nabodi de calcvlatoria nvmerorumque natura Sectiones quatuor. Ad Clarissimum virum Gasparvm Dovcivm Florentinvm, Caesareae Maiestatis Consiliarium. Coloniae Agrippinae, 1556.
   8 p.ℓ., 92 m, [95], numb.ℓ., 132, [1]p., 158ℓ. illus. diagrs. tables (fold.) 16 1/2 cm. (UM).
2. Logistice Regvlae Arithmeticae, qvam Algebram & Cossam uocant, compendio explicata.
   This is bound with (1).

LXXIX. W. P.(HILIPS).
The Pathway to Knowledge.
Written in Dutch and translated into English by W. P. London, 1596 (HN).

LXXX. JOHANN HEINRICH RAHN.
Teutsche Algebra oder Algebraische Rechenkunst usw. Zurich, 1659.
[12], 188, [8]p. diagrs., fold. tables. 19 1/2 cm. (UM).
Translated into English by BRANCKER (see Number XIII).

LXXXI. ROBERT RECORDE (1510-1558).
Whetstone of Witte. London, 1557.
336 p. tables (2 fold.) diagrs. 17 1/2 x 13 1/2 cm. (UM).

LXXXII. CARLO RENALDINI (RINALDINI) (1615-1698).
1. Opus Algebricum In quo praeter Communem, & antiquam Algebran Nova quoque pertractatur; atque fermissimis demonstrationibus ambae muniuntur. Ars analytica quam perobseurè Franciscus Vieta a literis mandauit satis, superq; declaratur Traditur...Anconae, 1644.
   6 p.ℓ., 290, [2]p. diagrs. 24 cm. (UM).
2. Opus Mathematicum in qvo Utraque Algebra, vetus scilicet, & noua a se in Opere, hac de re pridem edito, pertractata nouis praeceptis...Bononiae, 1655.
   12 p.ℓ., 475 p. illus., tables. 23 1/2 cm. (UM).
3. Ars. analytica mathematicum.
   a) pars Primum. Florence, 1665.
   b) pars Secundum. Patavii, 1669.
   3 v. front. (part.) diagrs. 35 cm. (UM).
4. De resolutione, & compositione mathematica. Patavia, 1668.
   4 p.ℓ., 225 (i.e., 535), 29 p. diagrs. 35 1/2 cm.
   There is another edition. Prima pars: Venice, 1684 (NYP).

LXXXIII. ANTICH ROCHA.
Arithmetica por Antich Rocha de Gerona compuesta, y de varios Auctores recopilada: Prouchosa para todos estados de gentes. Va anadido vn Compendio, para tener y regir los libros de Cuenta; traduzido de lingua Francesca en Romance Castellano. In Barcelona en casa de Claudio Bornat, a la Aguila fuerte. 1565. (Col. U).

LXXXIV. PETER ROTHE.
Peter Roth's Arithmetica philosophica oder Schöne neue und wohlgegründete uberauss künstliche Rechnung der Coss und Algebra von allerhand Reguln, Exempeln, AEquationem, Quaestionen, wunderkünstlichen und hochverständigen Fragstuchen. Nürnb., 1607 (LCU).
POGGENDORFF gives the date as 1600.

LXXXV. CHRISTOFF RUDOLFF.
Behend unnd Hubsch Rechnung durch die künstreichen regeln Algebre so gemeincklich die Coss genent werden, Strassburg, 1525. (See number XCIV, 3.)
RUDOLFF also published a collection of problems in several editions.

LXXXVI. BERNARDUS SALIGNACUS.
Arithmeticae libri duo, et Algebrae totidem, cum demonstrationibus. Francofurti. Apud Andream Wechelum. 1580.
4 p.ℓ., 56, 58-97, [99]-152, [2]p. 2 fold. tables. 20 x 15 1/2 cm. (UM).

LXXXVII. P. SCHAGEN.
Schagen Tel-Konst. Rotterdam, 1628 (Col. U).

LXXXVIII. JOHANN SCHEUBEL (1494-1570).
1. Euclidis Megarensis, Philosophi & Mathematici excellentissimi, sex libri Priores...Basileae, 1550.
4 p.ℓ., 315, [1]p. diagrs. 30 cm. (UM).
2. Algebrae Compendiosa Facilisque descriptio, qua depromuntur magna Arithmetices miracula. Authore Joanne Scheubelio Mathematicarum professore in academia Tubingensi. Parisiis, apud Gulielmum Cavellat, in Pingui Gallina, ex aduerso Collegii Cameracensis. 1551.
52. diagrs. 19 1/2 cm. (UM).
3. Compendivm arithmeticae artis, vt breuissimum ita longe utilissimum erudiendis tyronibus, non solium propter ordinen, quo pacis perstimgunter omnia huius artis capila:...Basileae, 1560.
1 p.ℓ., 193, 11 p., 1ℓ. illus. 16 cm. (UM).
Pr. ed., 1549 (GAP).

LXXXIX. FRANZ VAN SCHOOTEN (?-1661).
1. Exercitationum mathematicarum libri quinque. Lvgd. Batav. 1656-1657.
[12], 534, [2]p. diagrs. 22 cm. (UM).
There is a Dutch ed., 1660 (UM).
VAN SCHOOTEN edited DES CARTES' Geometry. (See Number XXVI.)
2. Mathematische oefeningen, begrepen in 5 boeken. Amsterdam G. van Goedesberg, 1660, (LMSA).

XC. JOHAN VAN SESEN.
Flores Arithmetici Edder Arithmetica (in 2 v.) Hamburg, 1622 (NYP).

XCI. JUAN MARTINEZ SILICIUS.
Arithmetica theoretica et practica. Paris, 1514.

XCII. JOHN STAMPIOEN (1610-?).
1. Resolutie ende ontbindinghe der twee vraeghstucken, in 1632 voor gestelt. Rotterdam, 1634 (LMSA).
2. Solutie op alle de quaestien, openbaer aangeslagen ende voor-gestelt door Ez. de Decker, waer bijgevoeght zijn d'ontbindinghe der twee fraegh-stucken in 1632 (LMSA) door dens. voor-gestelt enz. Rotterdam, 1634. (LMSA).

3. Openbaeringe der valscher practycken, gepleegt door Jac. a Waessenaer over het on-wis-constigh naebootsen der solutie, ghedsen door Johs. Stampioen de Jonghe in 1638. 's Gravenhage, 1638. (LMSA).
4. Algebra oft Nieuwe Stel-Regel, waer door alles ghevonden wordt inde Wis-Konst, wat vindtbaer is. Noyt voor de sen bekendt. Gevonden, ende beschreven Door Johan Stampioen d'Jonghe, Mathematicus. Residerende in's Graven-Hage. Matth:10. Want daer en is niet bedeckt, het welck niet en sal ontdeckt worden: ende verborgen, 't welck niet en sal gewêten worden. 's Graven-Hagen, Ghedruckt ten Huyse vanden Autheur in Sphaera Mundi. 1639.
   20 p.ℓ., 366, 28 p. diagrs. 19 1/2 cm. (UM).
5. Verclaringe over het gevoelen bij de professoren matheseos der Univ. tot Leyden uyt-gesproken nopende den regel fol. 25 van J. Stampioen ende 't ghene op de naem van een Wassenaer daer teghene is uytgecomen, enz. 's Gravenhage. 1640 (LMSA).
6. Den on-wiseen Wis-konstenaer J. J. Stampioenius ontdeckt door syne ongegrende weddinge inde mislucte solutien van syne eygene questien,...noch de solutien van twee swore geometrische questien door de Algebra...Leyden, 1656.
   BIERENS DE HAAN mentions many other works by this author.

XCIII. JOACH. STEGMANN.
Institutionum mathematicarum. Rakow, 1630.

XCIV. SIMON STEVIN.
1. Appendice algebraique contenant regle generale de toutes Equations. Leiden, 1594.
2. Wisconstighe gedachtenissen, inhoudende t' ghene daer hem in geoeffent heeft den doorluchtichsten Hoochgeboren Vorst ende Haere Maurits Prince van Oraengien, enz. (5 parts, 2 vol.) Leyden, J. Bouwens, 1605-1608. fo. (LMSA).
3. Les Oeuvres Mathematiques. Leyden, 1634.
   6 v. in 1. 4 p.ℓ., 222 p., 1ℓ., 678 p. illus., tables, diagrs. 35 1/2 cm. (UM).
4. L'Arithmetique contenant les computations, des nombres Arithmetiques ou vulgaires. Aussi l'Algebre avec les equations des cinq quantitez. Leyden, 1585; 4th ed., 1625.

XCV. MICHAEL STIFEL (1486-1567).
1. Arithmetica integra. Norimb., 1544.
2. Deutsche arithmetica. Norimb., 1545.
   4 p.ℓ., 92 numb.ℓ. tables. 19 1/2 x 15 cm. (UM).
3. Die Coss Christoph's, mit schönen Exemplen der Coss gebessert, Königsb., 1554. (UM).

XCVI. ANDRE TACQUET (1612-1660).
1. L'Arithmeticae theoria et Praxis, Auctore Andrea Tacquet Antverpiensi E. Societate Jesu, Matheseos Professor Lovanii, Apud Cyp. Coenestenivm, 1656.
   8 p.ℓ., 383, [5]p.
   H. BOSMANS says that this is a rare edition.
   It enjoyed four editions: 2d. ed., Anvers, 1665; 3rd ed., Anvers, 1682; 4th ed., Bruxelles, 1683.
   The University of Michigan has a copy of his Opera mathematica. 2 Tom in 1 vol. Antv., 1669. [60], 15, [4], 356, [6], 303 p. There is another ed., 1707.
2. Opera mathematica, demonstr. et propugn. a S. L. VETERANI. Antverp., Jac. Mersius, 1669 (LMSA).

XCVII. JOHN TAP (fl. 1596-1615).
The Pathway to Knowledge. London, 1613.

XCVIII. NICOLO TARTAGLIA.
1. La noua Scienti. 1550.
4 p.ℓ., 32 numb.ℓ. illus., diagrs. 4, 21 cm. (UM).
With this are bound his Qvesiti et inventione diverse. 1554; his Regola generale da svlevare. 1551; his Ragionamenti 1551. Ed., 1558 (UM).
2. La prima parte del general trattato di numeri, et misure di Nicolo Tartaglia ...Venice, 1556.
1 p.ℓ., 8, 277 numb.ℓ., illus., ports., tables, diagrs. 30 cm. (UM).
Subsequent editions of this work were: 1560; 1592-1593; Paris, 1578; Antwerp, 1578. He also published Tutte l'opera arithmetica, 1592 and L'arithmetique, Paris, 1613 (NYP), and works on EUCLID'S Elements.

XCIX. J. TWYSDEN (1607-1688).
Problematum Quorundam Mathematicorum, (De Triangulis tam Rectangulis quam Obliquangulis), Analytica Solvtio et Constructio. London, 1659.
1 p.ℓ., [12], 23, [1], 36, 8, 4, 17, [1], 27, [1], 4 p., [1], 10, 7 [1], 40, 16 p. 28 cm. (UM).
Bound in with SAMUEL FOSTER'S Miscellanies.

C. GIUSEPPE UNICORNO (1523-1610).
De l'arithmetica universale. Venetia, 1598 (UM).

CI. NICOLAUS REIMARUS URSUS (RAIMARI) (?-1600).
Ursi Dithmorsi arithmetica analytica vulgo Coss oder Algebra. Frankf. an der Oder, Barth. Voigt. 4,1601 (UM).

CII. I. L. SIEUR DE VAV-LEZARD.
Introduction en L'Art Analytic ou nouvelle Algebre De Francois Viete. Paris, 1630.
[8], 217, [2]p. diagrs. 16 1/2 cm. (UM).
He also wrote De utilitate mathematicarum artium. Venice. (GAP).

CIII. JUAN VENTALLOL.
La arismetica de Ivan Ventallol, traduzida de lengua Catalana en Castellana por el Doctor Juan Batista Tolra. Va anadido vn Tratado de la Arte Mayor llamada Algebra o Regla de la Cosa Compuesto por el mismo Doctor Tolra. En Tarragona, por Gabriel Roberto. 1619.
(Separate title page) Tratado de la Arte Mayor de Arismetica, llamado Algebra o Regla de la Cosa Compuesto por el Doctor JUAN BAPTISTA TOLRA. Tarragon...M. DC. XIX, 2 p. l., 79, (4) p. (UM).

CIV. FRANCOIS VIETE (1540-1603).
1. Opera mathematica. Lvge. Batav., 1646.
12 p.ℓ., 554 p. diagrs. 30 cm. (UM).
2. L'Algebra nouvelle...tradviete en francois par A. Vasset. Paris, 1630.
15 p.ℓ., 186, [2]p. diagrs. 21 1/2 cm. (UM).
3. De Aequationum Recognitione et Emendatione libro duo, publ. ALEXANDER ANDERSON. Paris, 1615.
4. Numerosa potestatum purarum atque adfectarum ad exegesin resolutione tractatus. Paris, 1660.
5. In Artem Analyticam Isagoge. Tours, 1591.
This is one of the earliest works on symbolic Algebra; to this an appendix called Logistice Speciosa was added, together with another called Zetetica.

CV. ANTONIO MARIA VISCONTI (latter part of 16th Century).
Practica Numerorum & mensurarum, ac alluuionis partitionem, inuestigandi, & ut in indice sequenti. Brixiae. Apud Iacobum, et Polycretum de Turlinis Fratres. 1581.
4 p.ℓ., 902 (i.e., 290) p., 1ℓ. diagrs. 20 1/2 x 16 cm. (UM).

CVI. JOANNES GERARDUS VOSSIUS (1577-1649).
1. De Universae mathesios natura et constitutione liber. Amsterdam, 1650.
1 p.ℓ., [14], 467, [35]p. 20 1/2 cm. (UM).
1660 edition can be found in the Library of the Mathematical Society at Amsterdam.
2. De quatuor artibus popularibus, de philologia et scientiis mathematicis, cui subj. chronologia mathematicorum, libri tres. Amstelaedami, J. Blaeu 1660. (LMSA).

CVII. JOHN WALLIS (1616-1703).
1. Mathesis universalis. Oxford, 1657.
2. Operum mathematicorum Oxford, 1657, 56.
4 p.ℓ., [16], 398 p., 1ℓ., 62, [5], 51, [6], 108, [8], 198, [4], 9 p. tables (fold.) diagrs. 19 1/2 cm. (UM).
3. A treatise of Algebra both historical and practical. London, 1685.
5 pts. in 1 v. diagrs. 33 cm. (UM).
4. Opera Mathematica, Oxford, 1693-1699, 3 vols. (UM).

CVIII. OTTO WESELLOW.
Flores arithmetici. Bremen, 1617.

CIX. WYANT VAN WESTERN.
Mathematische Vermaecklyckheden, te samen ghevoegt van verscheyden gheneuchlijcke ende boertige werck-stukken so uyt Arithmetica, Geometria, Astronomia, Geographia, Cosmographia, Musica, Physica, Optica, Catoptica, Architectonia, Scioterica, als uyt andere onghehoorde mysterien meer. Uyt het Fransch. (3 parts, 1 vol.) 2nd ed., Arnhem, Jacob van Biesen, 1641.
5th ed., Arnhem, Jacob van Biesen, 1672; and 6th ed., Amsterdam Wed. van Theunis Jacobsz. Lootsman, 1673. Another 6th ed., Amsterdam Mich. de Groot, 1673.

CX. MARTEN WILKEN (Martijn Wildens).
1. Flores Algebraici. Dat ist Algebra oder Coss...neben...audern theil...und Beschluss. Enbden, 1623.
2. Officina Algebrae. Waerinne door een nuraculosische Inventie seer konstelyck gewrocht worden d'AEquatien...Item de Vergelychingen...voorders Eene Extraordinarische Ontbindinghe...ten besluyt noch 300 deftige...Questien... Groeninghen, 1634.
This text was loaned by the University of Gronigen.
The University of Michigan library possesses photostats of the book.

CXI. EDMUND WINGATE.
Arithmetick made easie, edited by John Kersey. London, 1650. Algebra is found on p. 382.

CXII. GUILIELMUS XYLANDER (1532-1558).
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